On a conjecture of Montgomery-Vaughan on extreme values of automorphic L-functions at 1

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Abstract. In this paper, we prove a weaker form of a conjecture of Montgomery-Vaughan on extreme values of automorphic L-functions at 1.

Contents

§ 1. Introduction

The automorphic L-functions constitute a powerful tool for studying arithmetic, algebraic or geometric objects. For squarefree integer N and even integer k, denote by $\mathrm{H}_k^*(N)$ the set of all newforms of level N and of weight k. It is known that

$$|\mathrm{H}_k^*(N)| = \frac{k-1}{12}\varphi(N) + O\big((kN)^{2/3}\big), \tag{1.1}$$

where $\varphi(N)$ is the Euler function and the implied constant is absolute. Let $m \geqslant 1$ be an integer and let $L(s, \mathrm{sym}^m f)$ be the mth symmetric power L-function of $f \in \mathrm{H}_k^*(N)$ normalised so that the critical strip is given by $0 < \Re e\, s < 1$. The values of these functions at the edge of the critical strip contain information of great interest. For example, Serre [18] showed that the Sato-Tate conjecture is equivalent to $L(1 + i\tau, \mathrm{sym}^m f) \neq 0$ for all $m \in \mathbb{N}$ and $\tau \in \mathbb{R}$. The distribution of the values $L(1, \mathrm{sym}^m f)$ has received attention of many authors, including Goldfeld, Hoffstein & Lieman [2], Hoffstein & Lockhart [7], Luo [12], Royer [14, 15], Royer & Wu [16, 17], Cogdell & Michel [1], Habsieger & Royer [5] and Lau & Wu [10, 11]. In particular, Lau & Wu ([10], [11]) proved the following results:

(i) For every fixed integer $m \geqslant 1$, there are four positive constants $A_m^{\pm}$ and $B_m^{\pm}$ such that for any newform $f \in \mathrm{H}_k^*(1)$, under the Great Riemann Hypothesis (GRH) for $L(s, \mathrm{sym}^m f)$, we have, for $k \to \infty$,

$$\{1+o(1)\}(2B_m^-\log_2 k)^{-A_m^-} \leqslant L(1,\mathrm{sym}^m f) \leqslant \{1+o(1)\}(2B_m^+\log_2 k)^{A_m^+}. \tag{1.2}$$

Here (and in the sequel) $\log_j$ denotes the j-fold iterated logarithm. For most values of m, the constants $A_m^\pm$ and $B_m^\pm$ can be explicitly evaluated, for example,

$$\begin{cases} A_m^+ = m+1, & B_m^+ = e^\gamma \qquad (m\in\mathbb{N}),\\ A_m^- = m+1, & B_m^- = e^\gamma\zeta(2)^{-1} \qquad (\text{odd } m),\\ A_2^- = 1, & B_2^- = e^\gamma\zeta(2)^{-2},\\ A_4^- = \frac54, & B_4^- = e^\gamma B_4'^-, \end{cases}$$

where $\zeta(s)$ is the Riemann zeta-function, γ denotes the Euler constant and $B_4'^-$ is a positive constant given by a rather complicated Euler product ([10], Theorem 3).

(ii) In the opposite direction, it was shown unconditionally that for $m\in\{1,2,3,4\}$ there are newforms $f_m^\pm \in \mathrm{H}_k^*(1)$ such that for $k\to\infty$ ([10], Theorem 2),

$$\begin{cases} L(1,\mathrm{sym}^m f_m^+) \geqslant \{1+o(1)\}(B_m^+\log_2 k)^{A_m^+},\\ L(1,\mathrm{sym}^m f_m^-) \leqslant \{1+o(1)\}(B_m^-\log_2 k)^{-A_m^-}. \end{cases} \tag{1.3}$$

(iii) In the aim of removing GRH and closing up the gap coming from the factor 2 in (1.2) (comparing it with (1.3)), an almost all result was established. Let $\varepsilon>0$ be an arbitrarily small positive number, $m\in\{1,2,3,4\}$ and $2\mid k$. Then there is a subset E_k^* of $\mathrm{H}_k^*(1)$ such that $|\mathrm{E}_k^*| \ll \mathrm{H}_k^*(1)e^{-(\log k)^{1/2-\varepsilon}}$ and for each $f\in\mathrm{H}_k^*(1)\smallsetminus\mathrm{E}_k^*$, we have, for $k\to\infty$,

$$\{1+O(\varepsilon_k)\}(B_m^-\log_2 k)^{-A_m^-} \leqslant L(1,\mathrm{sym}^m f) \leqslant \{1+O(\varepsilon_k)\}(B_m^+\log_2 k)^{A_m^+}, \tag{1.4}$$

where $\varepsilon_k := (\log k)^{-\varepsilon}$ and the implied constants depend on ε only ([11], Corollary 2).

By comparing (1.3) with (1.4), the extreme values of $L(1,\mathrm{sym}^m f)$ seem to be given by (1.3). Clearly it is interesting to investigate further the size of exceptional set E_k^*. In the case of quadratic characters L-functions, Montgomery & Vaughan [13] proposed, based on a probabilistic model, three conjectures on the size of exceptional set. The first one has been proved recently by Granville & Soundararajan [4]. As Cogdell & Michel indicated in [1], it would be interesting to try to get, as close as possible, the analogues of the conjectures of Montgomery-Vaughan for automorphic L-functions. The analogue of Montgomery-Vaughan's first conjecture for the automorphic symmetric power L-functions can be stated as follows.

Conjecture. *Let $m\geqslant 1$ be a fixed integer and*

$$F_k(t,\mathrm{sym}^m) := \frac{1}{|\mathrm{H}_k^*(1)|}\sum_{f\in\mathrm{H}_k^*(1),\ L(1,\mathrm{sym}^m f)\geqslant (B_m^+t)^{A_m^+}} 1,$$

$$G_k(t,\mathrm{sym}^m) := \frac{1}{|\mathrm{H}_k^*(1)|}\sum_{f\in\mathrm{H}_k^*(1),\ L(1,\mathrm{sym}^m f)\leqslant (B_m^-t)^{-A_m^-}} 1.$$

Then there are positive constants $c_i=c_i(m)$ $(i=1,2)$ such that for $k\to\infty$,

$$\begin{cases} e^{-c_1(\log k)/\log_2 k} \ll F_k(\log_2 k,\mathrm{sym}^m) \ll e^{-c_2(\log k)/\log_2 k},\\ e^{-c_1(\log k)/\log_2 k} \ll G_k(\log_2 k,\mathrm{sym}^m) \ll e^{-c_2(\log k)/\log_2 k}. \end{cases} \tag{1.5}$$

The aim of this paper is to prove a weaker form of this conjecture for $m = 1$. In this case, we write, for simplification of notation,

$$L(s, f) = L(s, \mathrm{sym}^1 f), \qquad F_k(t) = F_k(t, \mathrm{sym}^1), \qquad G_k(t) = G_k(t, \mathrm{sym}^1).$$

In view of the trace formula of Petersson ([8], Theorem 3.6), it is more convenient to consider the weighted arithmetic distribution function. As usual, denote by

$$\omega_f := \frac{\Gamma(k-1)}{(4\pi)^{k-1}\|f\|}$$

the harmonic weight in modular forms theory and define the weighted arithmetic distribution functions

$$\widetilde{F}_k(t) := \Big(\sum_{f\in \mathrm{H}_k^*(1)} \omega_f\Big)^{-1} \sum_{f\in \mathrm{H}_k^*(1),\ L(1,f)\geqslant (e^\gamma t)^2} \omega_f,$$

$$\widetilde{G}_k(t) := \Big(\sum_{f\in \mathrm{H}_k^*(1)} \omega_f\Big)^{-1} \sum_{f\in \mathrm{H}_k^*(1),\ L(1,f)\leqslant (6\pi^{-2} e^\gamma t)^{-2}} \omega_f.$$

By using (1.1), the classical estimate

$$\sum_{f\in \mathrm{H}_k^*(1)} \omega_f = 1 + O\big(k^{-5/6}\big) \tag{1.6}$$

and the bound of Goldfeld, Hoffstein & Lieman [2]:

$$1/(k\log k) \ll \omega_f \ll (\log k)/k, \tag{1.7}$$

we easily see that

$$\begin{cases} \widetilde{F}_k(t)/\log k \ll F_k(t) \ll \widetilde{F}_k(t)\log k, \\ \widetilde{G}_k(t)/\log k \ll G_k(t) \ll \widetilde{G}_k(t)\log k. \end{cases} \tag{1.8}$$

This shows that in order to prove (1.5) it is sufficient to establish corresponding estimates of the same quality for $\widetilde{F}_k(t)$ and $\widetilde{G}_k(t)$.

Our main result is the following one.

Theorem 1. *For any $A \geqslant 1$ there are two positive constants $c = c(A)$ and $C = C(A)$ such that the estimate*

$$\widetilde{F}_k(t) = \{1 + \Delta_k(t)\}\exp\bigg\{-\frac{e^{t-\gamma_0}}{t}\bigg(1 + O\Big(\frac{1}{t}\Big)\bigg)\bigg\} \tag{1.9}$$

holds uniformly for $k \geqslant 16, 2 \mid k$ and $t \leqslant T(k)$, where γ_0 is given by (1.24) below, $|\theta| \leqslant 1$ and

$$\begin{cases} \Delta_k(t) := \theta e^{t-T(k)-C}(t/T(k))^{1/2} + O_A\big(e^{-ce^{t/5}} + (\log k)^{-A}\big), \\ T(k) := \log_2 k - \frac{5}{2}\log_3 k - \log_4 k - 3C. \end{cases} \tag{1.10}$$

In particular there are two positive constants c_1 and c_2 such that

$$e^{-c_1(\log k)/\{(\log_2 k)^{7/2}\log_3 k\}} \ll F_k(T(k)) \ll e^{-c_2(\log k)/\{(\log_2 k)^{7/2}\log_3 k\}}. \tag{1.11}$$

The similar estimates for $\widetilde{G}_k(t)$ and $G_k(T(k))$ hold also.

Remark 1. The estimates (1.11) of Theorem 1 can be considered as a weaker form of Montgomery-Vaughan's conjecture (1.5) for $m = 1$, since $T(k) \sim \log_2 k$ as $k \to \infty$. Moreover, if we could take $T(k) = \log_2 k$ in (1.11) then (1.9) would lead to the Montgomery-Vaughan's conjecture (1.5). Hence we fail from a shift

$$\frac{5}{2}\log_3 k + \log_4 k + 3C.$$

It seems however to be rather difficult to resolve completely this conjecture. One of the main difficulties is that there are no analogues of the quadratic reciprocity law and Graham-Ringrose's estimates for short characters sums of friable moduli [3], which have been exploited by Granville & Soundararajan [4].

In order to prove Theorem 1, we need to introduce a probabilistic model as in [1]. Consider a probability space (Ω, μ), with measure μ. Let $\mathrm{SU}(2)^\natural$ be the set of conjugacy classes of SU(2). The group SU(2) is endowed with its Haar measure μ_{H} and

$$\mathrm{SU}(2)^\natural = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0, \pi] \right\} \Big/ \sim$$

is endowed with the Sato-Tate measure $\mathrm{d}\mu_{\mathrm{st}}(\theta) := (2/\pi)\sin^2\theta\,\mathrm{d}\theta$, *i.e.,* the direct image of μ_{H} by the canonical projection $\mathrm{SU}(2) \to \mathrm{SU}(2)^\natural$. On the space (Ω, μ), define a sequence indexed by the prime numbers, $g^\natural(\omega) = \{g_p^\natural(\omega)\}_p$ of random matrices taking values in $\mathrm{SU}(2)^\natural$, given by

$$g_p^\natural(\omega) := \begin{pmatrix} e^{i\vartheta_p(\omega)} & 0 \\ 0 & e^{-i\vartheta_p(\omega)} \end{pmatrix}^\natural.$$

We assume that each function $g_p^\natural(\omega)$ is distributed according to the Sato-Tate measure. This means that, for each integrable function $\phi : \mathrm{SU}(2)^\natural \to \mathbb{R}$, the expected value of $\phi \circ g_p^\natural$ is

$$\mathbb{E}(\phi \circ g_p^\natural) := \int_\Omega \phi \circ g_p^\natural(\omega)\,\mathrm{d}\mu(\omega) = \int_0^\pi \phi\left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\right) \cdot (2/\pi)\sin^2\theta\,\mathrm{d}\theta.$$

Moreover, we assume that the sequence $g^\natural(\omega)$ is made of independent random variables. This means that, for any sequence of integrable functions $\{G_p : \mathrm{SU}(2)^\natural \to \mathbb{R}\}_p$, we have

$$\begin{aligned} \mathbb{E}\Big(\prod_p G_p \circ g_p^\natural\Big) &:= \int_\Omega \prod_p G_p \circ g_p^\natural(\omega)\,\mathrm{d}\mu(\omega) \\ &= \prod_p \int_\Omega G_p \circ g_p^\natural(\omega)\,\mathrm{d}\mu(\omega) \\ &= \prod_p \int_0^\pi G_p\left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\right) \cdot (2/\pi)\sin^2\theta\,\mathrm{d}\theta. \end{aligned} \tag{1.12}$$

Let I be the identity matrix. Then for $\Re e\, s > \frac{1}{2}$, the random Euler product

$$L(s, g^\natural(\omega)) := \prod_p \det\left(I - p^{-s} g_p^\natural(\omega)\right)^{-1} =: \prod_p L_p(s, g^\natural(\omega))$$

turns out to be absolutely convergent a.s.

Now we define our probabilistic distribution functions

$$\begin{cases} \Phi(t) := \mathrm{Prob}\big(\{L(1, g^\natural(\cdot)) \geqslant (e^\gamma t)^2\}\big), \\ \Psi(t) := \mathrm{Prob}\big(\{L(1, g^\natural(\cdot)) \leqslant (6\pi^{-2}e^\gamma t)^{-2}\}\big). \end{cases}$$

We shall prove Theorem 1 in two steps. The first one is to compare $\widetilde{F}_k(t)$ with $\Phi(t)$ (resp. $\widetilde{G}_k(t)$ with $\Psi(t)$).

Theorem 2. *For any $A \geqslant 1$ there are two positive constants $c = c(A)$ and $C = C(A)$ such that the asymptotic formulas*

$$\widetilde{F}_k(t) = \Phi(t)\{1 + \Delta_k(t)\} \qquad \text{and} \qquad \widetilde{G}_k(t) = \Psi(t)\{1 + \Delta_k(t)\} \tag{1.13}$$

hold uniformly for $k \geqslant 16, 2 \mid k$ and $t \leqslant T(k)$, where $\Delta_k(t)$ and $T(k)$ are defined by (1.10).

The second step of the proof of Theorem 1 is the evaluation of $\Phi(t)$ (resp. $\Psi(t)$). For this, we consider a truncated random Euler product

$$L(s, g^\natural(\omega); y) := \prod_{p \leqslant y} L_p(s, g^\natural(\omega))$$

and the corresponding distribution functions

$$\begin{cases} \Phi(t, y) := \mathrm{Prob}\big(\big\{L(1, g^\natural(\omega); y) \geqslant (e^\gamma t)^2\big\}\big), \\ \Psi(t, y) := \mathrm{Prob}\big(\big\{L(1, g^\natural(\omega); y) \leqslant (6\pi^{-2} e^\gamma t)^{-2}\big\}\big). \end{cases}$$

We have

$$\Phi(t) = \Phi(t, \infty) \qquad \text{and} \qquad \Psi(t) = \Psi(t, \infty). \tag{1.14}$$

We shall use the saddle-point method (introducted by Hildebrand & Tenenbaum [6]) to evaluate $\Phi(t, y)$ and $\Psi(t, y)$. For this, we need to introduce some notation. For $s \in \mathbb{C}$ and $y \geqslant 2$, define

$$E(s, y) := \mathbb{E}\big(L(1, g^\natural(\omega); y)^s\big) \qquad \text{and} \qquad E(s) := E(s, \infty), \tag{1.15}$$

where $\mathbb{E}(\cdot)$ denotes the expected value. We define also

$$\phi(s, y) := \log E(s, y), \qquad \phi_n(s, y) := \frac{\partial^n \phi}{\partial s^n}(s, y) \quad (n \geqslant 0). \tag{1.16}$$

According to Lemmas 2.3 and 8.1 below, there is an absolute constant $c \geqslant 2$ such that for $t \geqslant 4 \log c$ and $y \geqslant ce^t$, the equation

$$\phi_1(\kappa, y) = 2(\log t + \gamma) \tag{1.17}$$

has a unique positive solution $\kappa = \kappa(t, y)$ and for each integer $J \geqslant 1$, there are computable constants $\gamma_0, \gamma_1, \dots, \gamma_J$ such that the asymptotic formula

$$\kappa(t, y) = e^{t - \gamma_0}\bigg\{1 + \sum_{j=1}^{J} \frac{\gamma_j}{t^j} + O_J\bigg(\frac{1}{t^{J+1}} + \frac{e^t t}{y \log y}\bigg)\bigg\} \tag{1.18}$$

holds uniformly for $t \geqslant 1$ and $y \geqslant 2e^t$, the constant γ_0 beign given by (1.24) below.

Finally write $\sigma_n := \phi_n(\kappa, y)$.

Theorem 3. *We have*

$$\Phi(t,y)=\frac{E(\kappa,y)}{\kappa\sqrt{2\pi\sigma_2}(e^\gamma t)^{2\kappa}}\left\{1+O\left(\frac{t}{e^t}\right)\right\}$$

uniformly for $t\geqslant 1$ *and* $y\geqslant 2e^t$.

Theorem 4. *For each integer* $J\geqslant 1$, *we have*

$$\Phi(t,y)=\exp\left\{-\kappa\left[\sum_{j=1}^J\frac{a_j}{(\log\kappa)^j}+O_J\big(R_J(\kappa,y)\big)\right]\right\}\tag{1.19}$$

uniformly for $t\geqslant 1$ *and* $y\geqslant 2e^t$, *where the error term* $R_J(\kappa,y)$ *is given by*

$$R_J(\kappa,y):=\frac{1}{(\log\kappa)^{J+1}}+\frac{\kappa}{y\log y}\tag{1.20}$$

and

$$a_j:=\int_0^\infty\left(\frac{h(u)}{u}\right)'(\log u)^{j-1}\,\mathrm{d}u\tag{1.21}$$

with

$$h(u):=\begin{cases}\log\left(\dfrac{2}{\pi}\displaystyle\int_0^\pi e^{2u\cos\theta}\sin^2\theta\,\mathrm{d}\theta\right) & \text{if}\quad 0\leqslant u<1,\\[2ex] \log\left(\dfrac{2}{\pi}\displaystyle\int_0^\pi e^{2u\cos\theta}\sin^2\theta\,\mathrm{d}\theta\right)-2u & \text{if}\quad u\geqslant 1.\end{cases}\tag{1.22}$$

As a corollary of Theorem 4, we can obtain an asymptotic developpment for $\log\Phi(t,y)$ in t^{-1}. In particular we see that the probabilistic distribution function $\Phi(t)$ decays double exponentially as $t\to\infty$.

Corollary 5. *For each integer* $J\geqslant 1$, *there are computable constants* $a_1^*,\dots,a_J^*$ *such that the asymptotic formula*

$$\Phi(t,y)=\exp\left\{-e^{t-\gamma_0}\left[\sum_{j=1}^J\frac{a_j^*}{t^j}+O_J\big(R_J(e^t,y)\big)\right]\right\}\tag{1.23}$$

holds uniformly for $t\geqslant 1$ *and* $y\geqslant 2e^t$. *Further we have*

$$\gamma_0:=\frac12\int_0^\infty\frac{h'(u)}{u}\,\mathrm{d}u,\qquad a_1^*:=1,\qquad a_2^*:=\gamma_0-\frac{\gamma_0^2}{2}-\int_0^\infty\frac{h(u)}{u^2}(\log u)\,\mathrm{d}u.\tag{1.24}$$

In particular for each integer $J\geqslant 1$, *we have*

$$\Phi(t)=\exp\left\{-e^{t-\gamma_0}\left[\sum_{j=1}^J\frac{a_j^*}{t^j}+O_J\left(\frac{1}{t^{J+1}}\right)\right]\right\}\tag{1.25}$$

uniformly for $t\geqslant 1$.

Remark 2. (i) The same results hold also for $\Psi(t, y)$.

(ii) Taking $t = \log_2 k$ and $J = 1$ in (1.25) of Corollary 5, we see that the probabilistic distribution function $\Phi(t)$ (resp. $\Psi(t)$) verifies Montgomery-Vaughan's conjecture (1.5). But (1.13) is too weak to derive this conjecture for $F_k(t)$ (resp. $G_k(t)$). This means that we must take $T(k) = \log_2 k$ in Theorem 2, which seems be rather difficult.

(iii) Our method can be generalized (with a little extra effort) to prove that Theorems 1 and 2 hold for $L(1, \mathrm{sym}^m f)$ for $m \geqslant 1$ (unconditionally when $m = 1, 2, 3, 4$ and under Cogdell-Michel's hypothesis $\mathrm{Sym}^m(f)$ and $\mathrm{LSZ}^m(1)$ [1] when $m \geqslant 5$) and that Theorems 3, 4 and Corollary 5 are true for $L(1, \mathrm{sym}^m g^\natural(\omega); y)$ when $m \geqslant 1$.

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§ 2. Expression of $E(s, y)$ and existence of saddle-point

The aim of this section is to prove the existence of the saddle-point $\kappa(t, y)$, defined by equation (1.17). The first step is to give an explicite expression of $E(s, y)$, which is (1.24) of [1]. For the convenience of readers, we state it here as a lemma.

Lemma 2.1. *For prime p, real θ and complex number s, we define*

$$D_p(\theta) := \prod_{0 \leqslant j \leqslant 1} \left(1 - e^{i(1-2j)\theta} p^{-1}\right)^{-1} \quad \text{and} \quad E_p(s) := \frac{2}{\pi} \int_0^\pi D_p(\theta)^s \sin^2 \theta \,\mathrm{d}\theta. \tag{2.1}$$

Then for all $s \in \mathbb{C}$ and $y \geqslant 2$, we have

$$E(s, y) = \prod_{p \leqslant y} E_p(s). \tag{2.2}$$

Proof. Taking

$$G_p(M^\natural) = \begin{cases} \det\left(I - p^{-s'} M^\natural\right)^{-s} & \text{if } p \leqslant y \\ 1 & \text{otherwise} \end{cases}$$

in (1.12), we get

$$\begin{aligned}
\mathbb{E}\big(L(s', g^\natural(\omega); y)^s\big) &= \prod_{p \leqslant y} \mathbb{E}\big(L_p(s', g_p^\natural(\omega))^s\big) \\
&= \prod_{p \leqslant y} \int_\Omega \det\left(1 - p^{-s'} g_p^\natural(\omega)\right)^{-s} \mathrm{d}\mu(\omega) \\
&= \prod_{p \leqslant y} \frac{2}{\pi} \int_0^\pi \left(1 - 2p^{-s'} \cos\theta + p^{-2s'}\right)^{-s} \sin^2 \theta \,\mathrm{d}\theta.
\end{aligned}$$

Taking $s' = 1$ and noticing (1.15) and (2.1), we get the desired result. □

Lemma 2.2. *For all p and $\sigma > 0$, we have*

$$E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2 > 0.$$

In particular for all $\sigma > 0$ and $y \geqslant 2$, we have $\phi_2(\sigma, y) > 0$.

Proof. By using the definition (2.1) of $E_p(\sigma)$, it is easy to see that

$$\begin{aligned}E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2 &= \frac{4}{\pi^2}\int_0^\pi D_p(\theta)^\sigma \log^2 D_p(\theta)\sin^2\theta\,\mathrm{d}\theta \int_0^\pi D_p(\theta)^\sigma \sin^2\theta\,\mathrm{d}\theta \\ &\quad - \bigg(\frac{2}{\pi}\int_0^\pi D_p(\theta)^\sigma \log D_p(\theta)\sin^2\theta\,\mathrm{d}\theta\bigg)^2 \\ &= \frac{4}{\pi^2}\int_0^\pi\int_0^\pi D_p(\theta_1)^\sigma D_p(\theta_2)^\sigma\big(\log^2 D_p(\theta_1) - \log D_p(\theta_1)\log D_p(\theta_2)\big)\times \\ &\quad \times \sin^2\theta_1\sin^2\theta_2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2.\end{aligned}$$

In view of the symmetry in θ_1 and θ_2, the same formula holds if we exchange the roles of θ_1 and θ_2. Thus it follows that

$$E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2 = \frac{2}{\pi^2}\int_0^\pi\int_0^\pi D_p(\theta_1)^\sigma D_p(\theta_2)^\sigma \log^2\bigg(\frac{D_p(\theta_1)}{D_p(\theta_2)}\bigg)\sin^2\theta_1\sin^2\theta_2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2.$$

This proves the first assertion and the second follows immediately. □

Lemma 2.3. *There is an absolute constant $c \geqslant 2$ such that for $t \geqslant 4\log c$ and $y \geqslant ce^t$, the equation $\phi_1(\sigma, y) = 2(\log t + \gamma)$ has a unique positive solution in σ. Denoting by $\kappa(t, y)$ this solution, we have $\kappa(t, y) \asymp e^t$ uniformly for $t \geqslant 4\log c$ and $y \geqslant ce^t$.*

Proof. According to Lemma 4.3 below with the choice of $J = 1$, we have

$$\phi_1(\sigma, y) = 2(\log_2\sigma + \gamma) + O(1/\log\sigma)$$

for $y \geqslant \sigma \geqslant 2$. Thus

$$\begin{aligned}\phi(ce^t, y) &= 2\log(t + \log c) + 2\gamma + O\bigg(\frac{1}{t + \log c}\bigg) \\ &> 2\log t + 2\gamma\end{aligned}$$

and

$$\begin{aligned}\phi(c^{-1}e^t, y) &= 2\log(t - \log c) + 2\gamma + O\bigg(\frac{1}{t - \log c}\bigg) \\ &< 2\log t + 2\gamma,\end{aligned}$$

provided that c is a large constant and $t \geqslant 4\log c$. On the other hand, in view of Lemma 2.2, we know that for any $y \geqslant 2$, $\phi_1(\sigma, y)$ is an increasing function of σ in $(0, \infty)$. Hence the equation $\phi_1(\sigma, y) = 2(\log t + \gamma)$ has a unique positive solution $\kappa(t, y)$ and $c^{-1}e^t \leqslant \kappa(t, y) \leqslant ce^t$ for $t \geqslant 4\log c$ and $y \geqslant ce^t$. This completes the proof. □

§ 3. Preliminary lemmas

This section is devoted to establish some preliminary lemmas, which will be useful later.

Lemma 3.1. *Let $j \geqslant 0$ be a fixed real number. Then we have*

$$\int_0^{\pi} e^{2u\cos\theta}(1-\cos\theta)^j \sin^2\theta \,\mathrm{d}\theta \asymp_j e^{2u} u^{-(j+3/2)} \qquad (u \geqslant 1). \tag{3.1}$$

The implied constant depends on j only.

Proof. First we write

$$\begin{aligned}\int_0^{\pi} e^{2u\cos\theta}(1-\cos\theta)^j \sin^2\theta \,\mathrm{d}\theta &= \int_0^{\pi/2} \big(e^{2u\cos\theta}(1-\cos\theta)^j + e^{-2u\cos\theta}(1+\cos\theta)^j\big)\sin^2\theta\,\mathrm{d}\theta \\ &= \int_0^1 \big(e^{2ut}(1-t)^j + e^{-2ut}(1+t)^j\big)(1-t^2)^{1/2}\,\mathrm{d}t \\ &\asymp \int_0^1 e^{2ut}(1-t)^{j+1/2}\,\mathrm{d}t + \int_0^1 e^{-2ut}(1-t)^{1/2}\,\mathrm{d}t.\end{aligned}$$

By the change of variables $u(1-t) = v$, it follows that

$$\begin{aligned}\int_0^1 e^{2ut}(1-t)^{j+1/2}\,\mathrm{d}t &= e^{2u}u^{-(j+3/2)}\int_0^u e^{-2v}v^{j+1/2}\,\mathrm{d}v \\ &\asymp e^{2u}u^{-(j+3/2)}, \\ \int_0^1 e^{-2ut}(1-t)^{1/2}\,\mathrm{d}t &\leqslant \int_0^1 e^{-2ut}\,\mathrm{d}t \ll u^{-1}.\end{aligned}$$

We obtain the desired result by insertion of these estimates into the preceeding relation. □

Lemma 3.2. *Let $j \geqslant 0$ be an integer and*

$$E_{p,j}(\sigma) := \frac{2}{\pi}\int_0^{\pi} D_p(\theta)^{\sigma}(1-\cos\theta)^j\sin^2\theta\,\mathrm{d}\theta. \tag{3.2}$$

(In particular $E_{p,0}(\sigma) = E_p(\sigma)$.) Then we have

$$E_{p,j}(\sigma) = \frac{2^{j+3}}{\pi}\int_0^1 \bigg[\Big(1-\frac{1}{p}\Big)^2 + \frac{4u}{p}\bigg]^{-\sigma} u^{j+1/2}(1-u)^{1/2}\,\mathrm{d}u$$

and the estimate

$$E_{p,j}(\sigma)/E_p(\sigma) \ll (p/\sigma)^j \tag{3.3}$$

holds uniformly for all primes p and $\sigma > 0$. Further if $p \geqslant \sigma \geqslant 0$, we have

$$E_p(\sigma) \asymp 1. \tag{3.4}$$

The implied constant in (3.3) depends on j only and the one in (3.4) is absolute.

Proof. By the change of variables $u = \sin^2(\theta/2)$, a simple computation shows that the first assertion is true. Obviously (3.3) holds for $j = 0$.

Now assume that it is true for j. An integration by parts leads to

$$\begin{aligned}E_p(\sigma) &\gg_j \left(\frac{\sigma}{p}\right)^j \int_0^1 \left[\left(1-\frac{1}{p}\right)^2 + \frac{4u}{p}\right]^{-\sigma} u^{j+1/2}(1-u)^{1/2}\,\mathrm{d}u\\ &\gg_j \left(\frac{\sigma}{p}\right)^j \int_0^1 \left\{\left[\left(1-\frac{1}{p}\right)^2 + \frac{4u}{p}\right]^{-1}\frac{4\sigma}{p} + \frac{1}{2(1-u)}\right\}\times\\ &\qquad\times \left[\left(1-\frac{1}{p}\right)^2 + \frac{4u}{p}\right]^{-\sigma} u^{j+1+1/2}(1-u)^{1/2}\,\mathrm{d}u.\end{aligned}$$

On the other hand, we have

$$0<u<1 \;\;\Rightarrow\;\; \left[\left(1-\frac{1}{p}\right)^2 + \frac{4u}{p}\right]^{-1}\frac{4\sigma}{p} + \frac{1}{2(1-u)} \geqslant \left(1+\frac{1}{p}\right)^{-2}\frac{4\sigma}{p} \geqslant \frac{16\sigma}{9p}.$$

Inserting it into the preceeding estimate, we see that

$$\begin{aligned}E_p(\sigma) &\gg_j \left(\frac{\sigma}{p}\right)^{j+1} \int_0^1 \left[\left(1-\frac{1}{p}\right)^2 + \frac{4u}{p}\right]^{-\sigma} u^{j+1+1/2}(1-u)^{1/2}\,\mathrm{d}u\\ &\asymp_j \left(\frac{\sigma}{p}\right)^{j+1} E_{p,j+1}(\sigma).\end{aligned}$$

Thus (3.3) holds also for $j+1$.

Since $(1+1/p)^{-2} \leqslant D_p(\theta) \leqslant (1-1/p)^{-2}$ for all primes p and any $\theta \in \mathbb{R}$, we have $D_p(\theta)^\sigma \asymp 1$ uniformly for $p \geqslant \sigma \geqslant 0$ and $\theta \in \mathbb{R}$. This implies (3.4). □

Introduce the function

$$g(u) := \log\left(\frac{2}{\pi}\int_0^\pi e^{2u\cos\theta}\sin^2\theta\,\mathrm{d}\theta\right) \qquad (u \geqslant 0) \tag{3.5}$$

and let $h(u)$ be defined as in (1.22). Clearly we have

$$h(u) = \begin{cases} g(u) & \text{if } \; 0 \leqslant u < 1,\\ g(u) - 2u & \text{if } \; u \geqslant 1,\end{cases} \tag{3.6}$$

$$h'(u) = \begin{cases} g'(u) & \text{if } \; 0 \leqslant u < 1,\\ g'(u) - 2 & \text{if } \; u \geqslant 1,\end{cases} \tag{3.7}$$

$$h''(u) = g''(u) \quad (u \geqslant 0,\, u \neq 1). \tag{3.8}$$

Lemma 3.3. *We have*

$$h(u) \asymp \begin{cases} u^2 & \textit{if } \; 0 \leqslant u < 1,\\ \log(2u) & \textit{if } \; u \geqslant 1,\end{cases} \tag{3.9}$$

$$h'(u) \asymp \begin{cases} u & \textit{if } \; 0 \leqslant u < 1,\\ u^{-1} & \textit{if } \; u \geqslant 1,\end{cases} \tag{3.10}$$

$$h''(u) \asymp \begin{cases} 1 & \textit{if } \; 0 \leqslant u < 1,\\ u^{-2} & \textit{if } \; u \geqslant 1,\end{cases} \tag{3.11}$$

$$h'''(u) \asymp \begin{cases} u & \textit{if } \; 0 \leqslant u < 1,\\ u^{-3} & \textit{if } \; u \geqslant 1.\end{cases} \tag{3.12}$$

Proof. When $0 \leqslant u < 1$, we have

$$e^{2u\cos\theta} = \sum_{n=0}^{\infty} \frac{(u\cos\theta)^n}{n!}.$$

From this we deduce that

$$\begin{aligned} h(u) &= \log\bigg(\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{u^n}{n!}\int_0^{\pi}(\cos\theta)^n\sin^2\theta\,\mathrm{d}\theta\bigg) \\ &= \log\bigg(1+\sum_{\ell=1}^{\infty}\frac{2\cdot(2\ell-1)!!}{(2\ell)!(2\ell+2)!!}u^{2\ell}\bigg), \end{aligned} \tag{3.13}$$

where we have used the following facts:

$$\int_0^{\pi}(\cos\theta)^{2\ell+1}\sin^2\theta\,\mathrm{d}\theta = 0$$

and

$$\frac{2}{\pi}\int_0^{\pi}(\cos\theta)^{2\ell}\sin^2\theta\,\mathrm{d}\theta = \begin{cases} 1 & \text{if } \ \ell = 0, \\ 2\dfrac{(2\ell-1)!!}{(2\ell+2)!!} & \text{if } \ \ell \geqslant 1 \end{cases}$$

and where $n!!$ denotes the product of all positive integer from 1 to n having same parity than n. Now we easily deduce, from (3.13), the desired results (3.9)–(3.12) in the case of $0 \leqslant u < 1$.

The estimates of (3.9)–(3.12) for $u > 1$ are simple consequences of (3.1), by noticing the following relations

$$h'(u) = -2\frac{\displaystyle\int_0^{\pi} e^{2u\cos\theta}(1-\cos\theta)\sin^2\theta\,\mathrm{d}\theta}{\displaystyle\int_0^{\pi} e^{2u\cos\theta}\sin^2\theta\,\mathrm{d}\theta},$$

$$h''(u) = 4\frac{\displaystyle\int_0^{\pi} e^{2u\cos\theta}(1-\cos\theta)^2\sin^2\theta\,\mathrm{d}\theta}{\displaystyle\int_0^{\pi} e^{2u\cos\theta}\sin^2\theta\,\mathrm{d}\theta} - 4\left(\frac{\displaystyle\int_0^{\pi} e^{2u\cos\theta}(1-\cos\theta)\sin^2\theta\,\mathrm{d}\theta}{\displaystyle\int_0^{\pi} e^{2u\cos\theta}\sin^2\theta\,\mathrm{d}\theta}\right)^2.$$

This completes the proof. □

§ 4. Estimates of $\phi_n(\sigma, y)$

The aim of this section is to prove some estimates of $\phi_n(\sigma, y)$ for $n = 0, 1, 2, 3, 4$.

Lemma 4.1. *For any fixed integer $J \geqslant 1$, we have*

$$\phi_0(\sigma, y) = \sigma\bigg\{2\log_2\sigma + 2\gamma + \sum_{j=1}^{J}\frac{b_{j,0}}{(\log\sigma)^j} + O_J\big(R_J(\sigma, y)\big)\bigg\} \tag{4.1}$$

uniformly for $y \geqslant \sigma \geqslant 3$, where $R_J(\sigma, y)$ is defined as in (1.20) and

$$b_{j,0} := \int_0^{\infty}\frac{h(u)}{u^2}(\log u)^{j-1}\,\mathrm{d}u. \tag{4.2}$$

Proof. By the definition (2.1) of $D_p(\theta)$ and the one of $E_p(\sigma)$, it is easy to see that for $p \geqslant \sigma^{1/2}$, we have

$$D_p(\theta)^\sigma = e^{2(\sigma/p)\cos\theta}\bigg\{1 + O\bigg(\frac{\sigma}{p^2}\bigg)\bigg\}, \tag{4.3}$$

$$E_p(\sigma) = \bigg\{1 + O\bigg(\frac{\sigma}{p^2}\bigg)\bigg\}\frac{2}{\pi}\int_0^\pi e^{2(\sigma/p)\cos\theta}\sin^2\theta\,\mathrm{d}\theta. \tag{4.4}$$

From these, we deduce that

$$\sum_{\sigma^{1/2}<p\leqslant y} \log E_p(\sigma) = \sum_{\sigma^{1/2}<p\leqslant y} g(\sigma/p) + O(\sigma^{1/2}/\log\sigma) \tag{4.5}$$

where $g(u)$ is defined as in (3.5).

In order to treat the sum over $p \leqslant \sigma$, we write

$$E_p(\sigma) = (1 - 1/p)^{-2\sigma} E_p^*(\sigma),$$

where

$$E_p^*(\sigma) := \frac{2}{\pi}\int_0^\pi \bigg\{1 + \frac{2(1-\cos\theta)}{p}\bigg(1 - \frac{1}{p}\bigg)^{-2}\bigg\}^{-\sigma}\sin^2\theta\,\mathrm{d}\theta.$$

By using the change of variables $u = \sin^2(\theta/2)$, we have

$$\begin{aligned} E_p^*(\sigma) &= \frac{8}{\pi}\int_0^\pi \bigg\{1 + \frac{4}{p}\bigg(1 - \frac{1}{p}\bigg)^{-2}\sin^2(\theta/2)\bigg\}^{-\sigma}\sin^2(\theta/2)\cos^2(\theta/2)\,\mathrm{d}\theta \\ &\geqslant \frac{8}{\pi}\int_0^{p/2\sigma}\bigg\{1 + \frac{4}{p}\bigg(1 - \frac{1}{p}\bigg)^{-2} u\bigg\}^{-\sigma}\sqrt{u(1-u)}\,\mathrm{d}u \\ &\geqslant \frac{8}{\pi}\bigg(1 + \frac{8}{\sigma}\bigg)^{-\sigma}\int_0^{p/2\sigma}\sqrt{u(1-u)}\,\mathrm{d}u \\ &\geqslant C\bigg(\frac{p}{\sigma}\bigg)^{3/2}, \end{aligned}$$

where $C > 0$ is a constant. On the other hand, we have trivially $E_p^*(\sigma) \leqslant 1$ for all p and $\sigma > 0$. Thus $|\log E_p^*(\sigma)| \ll \log(\sigma/p)$ for $p \leqslant \sigma^{1/2}$ and

$$\sum_{p\leqslant\sigma^{1/2}} \big|\log E_p^*(\sigma)\big| \ll \sum_{p\leqslant\sigma^{1/2}} \log(\sigma/p) \ll \sigma^{1/2}. \tag{4.6}$$

Combining (4.5) and (4.6), we can write

$$\sum_{p\leqslant y} \log E_p(\sigma) = 2\sigma \sum_{p\leqslant\sigma^{1/2}} \log(1 - 1/p)^{-1} + \sum_{\sigma^{1/2}<p\leqslant y} g(\sigma/p) + O(\sigma^{1/2}).$$

In view of (3.6) and the following estimate

$$\sum_{\sigma^{1/2}<p\leqslant\sigma} \big(2\sigma\log(1 - 1/p)^{-1} - 2\sigma/p\big) \ll \sum_{\sigma^{1/2}<p\leqslant\sigma} \sigma/p^2 \ll \sigma^{1/2}/\log\sigma,$$

the preceeding estimate can be written as

$$\sum_{p\leqslant y} \log E_p(\sigma) = 2\sigma \sum_{p\leqslant\sigma} \log(1 - 1/p)^{-1} + \sum_{\sigma^{1/2}<p\leqslant y} h(\sigma/p) + O(\sigma^{1/2}). \tag{4.7}$$

By using the prime number theorem in the form

$$\pi(t) := \sum_{p\leqslant t} 1 = \int_2^t \frac{\mathrm{d}v}{\log v} + O\Big(te^{-8\sqrt{\log t}}\Big), \tag{4.8}$$

it follows that

$$\sum_{\sigma^{1/2}<p\leqslant y} h\Big(\frac{\sigma}{p}\Big) = \int_{\sigma^{1/2}}^y \frac{h(\sigma/t)}{\log t}\,\mathrm{d}t + O(R_0), \tag{4.9}$$

where

$$\begin{aligned} R_0 &:= h\Big(\frac{\sigma}{y}\Big)ye^{-8\sqrt{\log y}} + h\big(\sigma^{1/2}\big)\sigma^{1/2}e^{-4\sqrt{\log\sigma}} + \int_{\sigma^{1/2}}^y (\sigma/t)|h'(\sigma/t)|e^{-8\sqrt{\log t}}\,\mathrm{d}t \\ &\ll \frac{\sigma^2}{y}e^{-8\sqrt{\log y}} + \sigma^{1/2}e^{-2\sqrt{\log\sigma}} + \int_{\sigma^{1/2}}^{\sigma} e^{-2\sqrt{\log t}}\,\mathrm{d}t + \sigma^2\int_\sigma^y \frac{e^{-8\sqrt{\log t}}}{t^2}\,\mathrm{d}t \\ &\ll \sigma e^{-\sqrt{\log\sigma}} \end{aligned}$$

by use of Lemma 3.3.

In order to evaluate the integral of (4.9), we use the change of variables $u = \sigma/t$ to write

$$\begin{aligned} \int_{\sigma^{1/2}}^y \frac{h(\sigma/t)}{\log t}\,\mathrm{d}t &= \sigma\int_{\sigma/y}^{\sigma^{1/2}} \frac{h(u)}{u^2\log(\sigma/u)}\,\mathrm{d}u \\ &= \sigma\int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2\log(\sigma/u)}\,\mathrm{d}u + O\big(R_0'\big) \end{aligned}$$

where

$$\begin{aligned} R_0' &:= \sigma\int_0^{\sigma/y} \frac{|h(u)|}{u^2\log(\sigma/u)}\,\mathrm{d}u + \sigma\int_0^{\sigma^{-1/2}} \frac{|h(u)|}{u^2\log(\sigma/u)}\,\mathrm{d}u \\ &\ll \frac{\sigma^2}{y\log y} + \frac{\sigma^{1/2}}{\log\sigma}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2\log(\sigma/u)}\,\mathrm{d}u &= \frac{1}{\log\sigma}\int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2(1-(\log u)/\log\sigma)}\,\mathrm{d}u \\ &= \sum_{j=1}^J \frac{1}{(\log\sigma)^j}\int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2}(\log u)^{j-1}\,\mathrm{d}u + O\bigg(\frac{1}{(\log\sigma)^{J+1}}\bigg). \end{aligned}$$

Extending the interval of integration $[\sigma^{-1/2}, \sigma^{1/2}]$ to $(0,\infty)$ and bounding the contributions of $(0,\sigma^{-1/2}]$ and $[\sigma^{1/2},\infty)$ by using (3.9) of Lemma 3.3, we have

$$\int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2}(\log u)^{j-1}\,\mathrm{d}u = b_{j,0} + O\bigg(\frac{(\log\sigma)^j}{\sigma^{1/2}}\bigg).$$

Combining these estimates, we find that

$$\sum_{\sigma^{1/2}<p\leqslant y} h\Big(\frac{\sigma}{p}\Big) = \sigma\bigg\{\sum_{j=1}^J \frac{b_{j,0}}{(\log\sigma)^j} + O_J\big(R_J(\sigma,y)\big)\bigg\}. \tag{4.10}$$

Now the desired result follows from (4.7), (4.10) and the prime number theorem in the form

$$\sum_{p\leqslant\sigma}\log(1-1/p)^{-1}=\log_2\sigma+\gamma+O\Big(e^{-2\sqrt{\log\sigma}}\Big). \tag{4.11}$$

This completes the proof. □

Remark 3. In view of (1.3), we can write (4.1) as

$$\phi_0(\sigma,y)=\sigma\bigg\{\log(B_1^+\log\sigma)^{A_1^+}+\sum_{j=1}^J\frac{b_{j,0}}{(\log\sigma)^j}+O_J\big(R_J(\sigma,y)\big)\bigg\}$$

uniformly for $y\geqslant\sigma\geqslant3$. In the case $\sigma<0$, a similar asymptotic formula (with A_1^-,B_1^- and corresponding $b_{j,0}^-$ in place of A_1^+,B_1^+ and $b_{j,0}$) can be established uniformly for $y\geqslant-\sigma\geqslant3$. As indicated in the introduction, Lemma 4.1 can be easily generalised to the general case $m\geqslant1$. Thus we give an improvement and generalisation of Corollaries A and C of [15], of Theorem B of [5], and an improvement of Theorem 1.12 of [1]. It is worthy to indicate that our method seems to be simpler and more natural.

Lemma 4.2. *We have*

$$\frac{E_p'(\sigma)}{E_p(\sigma)}=\begin{cases}\log D_p(0)+O\Big(\dfrac1\sigma\Big) & \text{for all } p \text{ and } \sigma>0,\\[2ex] \dfrac12 g'\Big(\dfrac\sigma p\Big)\log D_p(0)+O\Big(\dfrac1{p^2}+\dfrac\sigma{p^3}\Big) & \text{if}\quad p\geqslant\sigma^{1/2},\end{cases} \tag{4.12}$$

where $g(u)$ is defined as in (3.5).

Proof. First we write

$$\begin{aligned}E_p'(\sigma)&=\frac2\pi\int_0^\pi D_p(\theta)^\sigma\log D_p(\theta)\sin^2\theta\,\mathrm d\theta\\&=E_p(\sigma)\log D_p(0)+R',\end{aligned} \tag{4.13}$$

where

$$R':=\frac2\pi\int_0^\pi D_p(\theta)^\sigma\log\bigg(\frac{D_p(\theta)}{D_p(0)}\bigg)\sin^2\theta\,\mathrm d\theta. \tag{4.14}$$

Since

$$\bigg|\log\bigg(\frac{D_p(\theta)}{D_p(0)}\bigg)\bigg|=\bigg|-\log\bigg(1+\frac{2p(1-\cos\theta)}{(p-1)^2}\bigg)\bigg|\leqslant\frac{2p(1-\cos\theta)}{(p-1)^2}\leqslant\frac{8(1-\cos\theta)}{p},$$

it follows from (3.3) of Lemma 3.2 with $j=1$ that

$$\frac{R'}{E_p(\sigma)}\ll\frac{E_{p,1}(\sigma)}{pE_p(\sigma)}\ll\frac1\sigma$$

for all p and $\sigma>0$. This implies, via (4.13), the first estimate of (4.12).

We have

$$\begin{aligned}\log D_p(\theta)&=(\cos\theta)(2/p)+O\big(1/p^2\big)\\&=(\cos\theta)\log D_p(0)+O\big(1/p^2\big).\end{aligned}$$

Inserting it and (4.3) into the first relation of (4.13) and in view of (4.4), we can write, for $p \geqslant \sigma^{1/2}$,

$$\begin{aligned} E_p'(\sigma) &= \bigg\{1+O\bigg(\frac{\sigma}{p^2}\bigg)\bigg\}\frac{2}{\pi}\int_0^\pi e^{2(\sigma/p)\cos\theta}\bigg\{(\cos\theta)\log D_p(0)+O\bigg(\frac{1}{p^2}\bigg)\bigg\}\sin^2\theta\,\mathrm{d}\theta \\ &= \bigg\{1+O\bigg(\frac{\sigma}{p^2}\bigg)\bigg\}\frac{2}{\pi}\int_0^\pi e^{2(\sigma/p)\cos\theta}(\cos\theta)\sin^2\theta\,\mathrm{d}\theta\log D_p(0)+O\bigg(\frac{E_p(\sigma)}{p^2}\bigg). \end{aligned}$$

From this and (4.4), we deduce

$$\frac{E_p'(\sigma)}{E_p(\sigma)} = \bigg\{1+O\bigg(\frac{\sigma}{p^2}\bigg)\bigg\}\frac{1}{2}g'\bigg(\frac{\sigma}{p}\bigg)\log D_p(0)+O\bigg(\frac{1}{p^2}\bigg),$$

which implies the second estimate of (4.12). This completes the proof. □

Lemma 4.3. *Let $J \geqslant 1$ be a fixed integer. Then we have*

$$\phi_1(\sigma,y) = 2\log_2\sigma + 2\gamma + \sum_{j=1}^J \frac{b_{j,1}}{(\log\sigma)^j} + O_J\big(R_J(\sigma,y)\big)$$

uniformly for $y \geqslant \sigma \geqslant 3$, where the constant $b_{j,1}$ is given by

$$b_{j,1} := \int_0^\infty \frac{h'(u)}{u}(\log u)^{j-1}\,\mathrm{d}u \tag{4.15}$$

and $R_J(\sigma,y)$ is defined as in (1.20).

Proof. We have

$$\phi_1(\sigma,y) = \sum_{p\leqslant y} E_p'(\sigma)/E_p(\sigma).$$

Using the first relation of (4.12) for $p \leqslant \sigma^{2/3}$ and the second for $\sigma^{2/3} < p \leqslant y$, we obtain

$$\phi_1(\sigma,y) = \sum_{p\leqslant\sigma^{2/3}} \log D_p(0) + \frac{1}{2}\sum_{\sigma^{2/3}<p\leqslant y} g'\bigg(\frac{\sigma}{p}\bigg)\log D_p(0) + O\bigg(\frac{1}{\sigma^{1/3}}\bigg).$$

In view of (3.7), the preceeding formula can be written as

$$\phi_1(\sigma,y) = \sum_{p\leqslant\sigma}\log D_p(0) + \sum_{\sigma^{2/3}<p\leqslant y} h'\bigg(\frac{\sigma}{p}\bigg)\log\bigg(1-\frac{1}{p}\bigg)^{-1} + O\bigg(\frac{1}{\sigma^{1/3}}\bigg). \tag{4.16}$$

Similarly to (4.10), we can prove that

$$\sum_{\sigma^{2/3}<p\leqslant y} h'\bigg(\frac{\sigma}{p}\bigg)\log\bigg(1-\frac{1}{p}\bigg)^{-1} = \sum_{j=1}^J \frac{b_{j,1}}{(\log\sigma)^j} + O_J\big(R_J(\sigma,y)\big), \tag{4.17}$$

using (3.10), (3.11) and (4.11) instead of (3.9), (3.10) and (4.8). Now the desired result follows from (4.16), (4.10) and (4.17). □

Lemma 4.4. *We have*

$$\frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \begin{cases} O\Big(\dfrac{1}{\sigma^2}\Big) & \text{if} \quad p \leqslant \sigma^{1/2}, \\ \dfrac{1}{p^2} g''\Big(\dfrac{\sigma}{p}\Big) + O\bigg(\min\Big\{\dfrac{1}{\sigma^2 p}, \dfrac{1}{\sigma p^2}\Big\}\bigg) & \text{if} \quad p > \sigma^{1/2}, \end{cases} \tag{4.18}$$

where $g(u)$ is defined as in (3.5).

Proof. First we write

$$\begin{aligned} E_p''(\sigma) &= \frac{2}{\pi}\int_0^\pi D_p(\theta)^\sigma \log^2 D_p(\theta) \sin^2\theta \,\mathrm{d}\theta \\ &= E_p(\sigma)\log^2 D_p(0) + R'', \end{aligned} \tag{4.19}$$

where

$$R'' := \frac{2}{\pi}\int_0^\pi D_p(\theta)^\sigma \Big(\log^2 D_p(\theta) - \log^2 D_p(0)\Big)\sin^2\theta \,\mathrm{d}\theta.$$

Using (4.13) and (4.19), we can deduce

$$\frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \frac{R'' - 2R'\log D_p(0)}{E_p(\sigma)} - \left(\frac{R'}{E_p(\sigma)}\right)^2, \tag{4.20}$$

where R' is defined as in (4.14).

From the definitions of R' and R'', a simple calculation shows that

$$R'' - 2R'\log D_p(0) = \frac{2}{\pi}\int_0^\pi D_p(\theta)^\sigma \log^2\left(\frac{D_p(\theta)}{D_p(0)}\right)\sin^2\theta \,\mathrm{d}\theta.$$

Since

$$\log^2\left(\frac{D_p(\theta)}{D_p(0)}\right) = \log^2\left(1 + \frac{2p(1-\cos\theta)}{(p-1)^2}\right) = \frac{4(1-\cos\theta)^2}{p^2} + O\bigg(\frac{(1-\cos\theta)^2}{p^3}\bigg),$$

we have

$$R'' - 2R'\log D_p(0) = \frac{4}{p^2}E_{p,2}(\sigma) + O\bigg(\frac{E_{p,2}(\sigma)}{p^3}\bigg),$$

where $E_{p,j}(\sigma)$ is defined as in (3.2). By using (3.3) with the choice of $j = 2$ and the trivial estimate $E_{p,2}(\sigma) \leqslant 4E_p(\sigma)$, we deduce

$$\frac{R'' - 2R'\log D_p(0)}{E_p(\sigma)} = \frac{4}{p^2}\frac{E_{p,2}(\sigma)}{E_p(\sigma)} + O\bigg(\min\Big\{\frac{1}{\sigma^2 p}, \frac{1}{p^3}\Big\}\bigg). \tag{4.21}$$

Similarly we have

$$\log\left(\frac{D_p(\theta)}{D_p(0)}\right) = -\log\left(1 + \frac{2p(1-\cos\theta)}{(p-1)^2}\right) = -\frac{2(1-\cos\theta)}{p} + O\bigg(\frac{(1-\cos\theta)}{p^2}\bigg),$$

and therefore

$$R' = -\frac{2}{p}E_{p,1}(\sigma) + O\bigg(\frac{E_{p,1}(\sigma)}{p^2}\bigg).$$

Now (3.3) with $j = 1$ and the trivial estimate $E_{p,1}(\sigma) \leqslant 2E_p(\sigma)$ imply

$$\begin{aligned}\left(\frac{R'}{E_p(\sigma)}\right)^2 &= \frac{4}{p^2}\left(\frac{E_{p,1}(\sigma)}{E_p(\sigma)}\right)^2 + O\left(\frac{E_{p,1}(\sigma)^2}{p^3 E_p(\sigma)^2}\right) \\ &= \frac{4}{p^2}\left(\frac{E_{p,1}(\sigma)}{E_p(\sigma)}\right)^2 + O\left(\min\left\{\frac{1}{\sigma^2 p}, \frac{1}{p^3}\right\}\right).\end{aligned} \tag{4.22}$$

Inserting (4.21) and (4.22) into (4.20) and in view of (4.14), we deduce

$$\frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \frac{4}{p^2}h_p(\sigma) + O\left(\min\left\{\frac{1}{\sigma^2 p}, \frac{1}{p^3}\right\}\right) \tag{4.23}$$

for all p and $\sigma > 0$, where

$$h_p(\sigma) := \frac{E_{p,2}(\sigma)}{E_p(\sigma)} - \left(\frac{E_{p,1}(\sigma)}{E_p(\sigma)}\right)^2.$$

When $p \leqslant \sigma^{1/2}$, the inequality (3.3) of Lemma 3.2 implies that $h_p(\sigma) \ll (p/\sigma)^2$. From this and (4.23) we deduce the first estimate of (4.18).

If $p \geqslant \sigma^{1/2}$, we can use (4.3), (3.11) and (3.8) to write

$$\begin{aligned}4h_p(\sigma) &= g''\left(\frac{\sigma}{p}\right)\left\{1 + O\left(\frac{\sigma}{p^2}\right)\right\} \\ &= g''\left(\frac{\sigma}{p}\right) + O\left(\min\left\{\frac{\sigma}{p^2}, \frac{1}{\sigma}\right\}\right).\end{aligned}$$

Inserting it into (4.23) and in view of Lemma 3.1, we get, for $p \geqslant \sigma^{1/2}$,

$$\frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \frac{1}{p^2}g''\left(\frac{\sigma}{p}\right) + O\left(\min\left\{\frac{1}{\sigma^2 p}, \frac{1}{\sigma p^2}\right\}\right).$$

This completes the proof. □

Lemma 4.5. *Let $J \geqslant 1$ be a fixed integer. Then we have*

$$\phi_2(\sigma, y) = \frac{1}{\sigma}\left\{\sum_{j=1}^{J} \frac{b_{j,2}}{(\log \sigma)^j} + O_J\big(R_J(\sigma, y)\big)\right\}$$

uniformly for $y \geqslant \sigma \geqslant 2$, where

$$b_{j,2} := \int_0^\infty h''(u)(\log u)^{j-1}\,\mathrm{d}u.$$

In particular $b_{1,2} = 2$.

Proof. From Lemma 4.4 and (3.8), we deduce easily that

$$\begin{aligned}\phi_2(\sigma, y) &= \sum_{p \leqslant y} \frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} \\ &= \sum_{\sigma^{1/2} < p \leqslant y} \frac{g''(\sigma/p)}{p^2} + O\left(\frac{1}{\sigma^{3/2}\log \sigma}\right) \\ &= \sum_{\sigma^{1/2} < p \leqslant y} \frac{h''(\sigma/p)}{p^2} + O\left(\frac{1}{\sigma^{3/2}\log \sigma}\right).\end{aligned}$$

Similarly to (4.10), we can prove that

$$\sum_{\sigma^{1/2}<p\leqslant y}\frac{h''(\sigma/p)}{p^2}=\frac{1}{\sigma}\bigg\{\sum_{j=1}^{J}\frac{b_{j,2}}{(\log\sigma)^j}+O_J\big(R_J(\sigma,y)\big)\bigg\},$$

by using (3.11), (3.12) and (4.8). Now the desired result follows from the preceeding two estimates.

Finally

$$\begin{aligned}b_{1,2}&=\int_0^1 h''(u)\,\mathrm{d}u+\int_1^\infty h''(u)\,\mathrm{d}u\\&=h'(1-)-h'(1+)=h'(1-)-\big(h'(1-)-2\big)=2.\end{aligned}$$

This completes the proof. □

Similarly (even more easily, since we only need an upper bound instead of an asymptotic formula), we can prove the following result.

Lemma 4.6. *We have*

$$\phi_n(\sigma,y)\ll 1/(\sigma^{n-1}\log\sigma)\qquad(n=3,4)\tag{4.24}$$

uniformly for $y\geqslant\sigma\geqslant 3$.

§ 5. Estimate of $|E(\kappa+i\tau,y)|$

Lemma 5.1. *For any $\delta\in(0,\frac14)$, there are two absolute positive constants c_1,c_2 and a positive constant $c_3=c_3(\delta)$ such that for all $y\geqslant\sigma\geqslant 3$ we have*

$$\left|\frac{E(\sigma+i\tau,y)}{E(\sigma,y)}\right|\leqslant\begin{cases}1 & \textit{if}\ \ |\tau|\leqslant c_1\sigma^{1/2}\log\sigma\ \ \textit{or}\ \ |\tau|\geqslant y^{1/\delta},\\ e^{-c_2\tau^2/[\sigma(\log\sigma)^2]} & \textit{if}\ \ c_1\sigma^{1/2}\log\sigma\leqslant|\tau|\leqslant\sigma,\\ e^{-c_3|\tau|^\delta} & \textit{if}\ \ \sigma\leqslant|\tau|\leqslant y^{1/\delta}.\end{cases}\tag{5.1}$$

Proof. First we write

$$\begin{aligned}E_p(s)&=\frac{2}{\pi}\int_0^\pi\big(D_p(\theta)^{-1}\big)^{-s}\sin^2\theta\,\mathrm{d}\theta\\&=\frac{2}{\pi}\int_0^\pi\frac{\sin^2\theta}{(1-s)(D_p(\theta)^{-1})'}\,\mathrm{d}\big(D_p(\theta)^{-1}\big)^{1-s}.\end{aligned}$$

Since $(D_p(\theta)^{-1})'=2p^{-1}\sin\theta$, after a simplification and an integration by parts it follows that

$$\begin{aligned}E_p(s)&=\frac{p}{\pi(s-1)}\int_0^\pi D_p(\theta)^{s-1}\cos\theta\,\mathrm{d}\theta\\&=\frac{p}{\pi(s-1)}\int_0^{\pi/2}\big\{D_p(\theta)^{s-1}-D_p(\pi-\theta)^{s-1}\big\}\cos\theta\,\mathrm{d}\theta.\end{aligned}$$

This implies that

$$\left|\frac{E_p(s)}{E_p(\sigma)}\right|=\left|\frac{\sigma-1}{s-1}\right|\left|\frac{E_p^*(s)}{E_p^*(\sigma)}\right|\tag{5.2}$$

with

$$E_p^*(s) := \int_0^{\pi/2} \big\{D_p(\theta)^{s-1} - D_p(\pi-\theta)^{s-1}\big\}\cos\theta \,\mathrm{d}\theta.$$

1° Case of $\sigma^{1/\delta} < |\tau| \leqslant y^{1/\delta}$

Write

$$E_p^*(s) = \int_0^{\pi/2} D_p(\theta)^{s-1}\big\{1 - \Delta_p(\theta)^{s-1}\big\}\cos\theta \,\mathrm{d}\theta$$

with

$$\Delta_p(\theta) := \frac{1-2p^{-1}\cos\theta + p^{-2}}{1+2p^{-1}\cos\theta+p^{-2}}.$$

It is clear that for all p, the function $\theta \mapsto \Delta_p(\theta)$ is increasing on $[0, \pi/2]$. It follows that

$$\begin{aligned} E_p^*(\sigma) &\geqslant \int_0^{\pi/4} D_p(\theta)^{\sigma-1}\big\{1-\Delta_p(\theta)^{\sigma-1}\big\}\cos\theta\,\mathrm{d}\theta \\ &\geqslant \big\{1-\Delta_p(\pi/4)^{\sigma-1}\big\}\int_0^{\pi/4} D_p(\theta)^{\sigma-1}\cos\theta\,\mathrm{d}\theta \end{aligned}$$

for all p and $\sigma \geqslant 1$. This implies that

$$\left|\frac{1}{E_p^*(\sigma)}\int_0^{\pi/4} D_p(\theta)^{\sigma-1}\cos\theta\,\mathrm{d}\theta\right| \leqslant \frac{1}{1-\Delta_p(\pi/4)^{\sigma-1}}. \tag{5.3}$$

Similarly since the function $\theta \mapsto D_p(\theta)^{\sigma-1}\cos\theta$ is decreasing on $[0, \pi/2]$ for all p and $\sigma \geqslant 2$, we can deduce, via (5.3), that

$$\left|\frac{1}{E_p^*(\sigma)}\int_{\pi/4}^{\pi/2} D_p(\theta)^{\sigma-1}\cos\theta\,\mathrm{d}\theta\right| \leqslant \frac{1}{1-\Delta_p(\pi/4)^{\sigma-1}}. \tag{5.4}$$

From (5.3) and (5.4), we deduce that

$$\left|\frac{E_p^*(s)}{E_p^*(\sigma)}\right| \leqslant \frac{2}{1-\Delta_p(\pi/4)^{\sigma-1}}.$$

It is easy to verify that for all $p \geqslant \sigma \geqslant 2$, we have

$$\Delta_p\Big(\frac{\pi}{4}\Big)^{\sigma-1} \leqslant \Big(1-\frac{\sqrt{2}}{p}+\frac{1}{p^2}\Big)^{\sigma-1} \leqslant 1-\frac{\sigma-1}{4p}.$$

Combining these estimates with (5.2), we obtain

$$\left|\frac{E_p(s)}{E_p(\sigma)}\right| \leqslant \frac{8p}{|s-1|} \leqslant \frac{p^4}{|\tau|} \qquad (p \geqslant \sigma).$$

By multiplying this inequality for $\sigma < p \leqslant |\tau|^\delta$ ($\leqslant y$) and the trivial inequality $|E_p(s)| \leqslant |E_p(\sigma)|$ for the others p, we deduce, via the prime number theorem, that

$$\begin{aligned} \left|\frac{E(s,y)}{E(\sigma,y)}\right| &\leqslant \exp\bigg\{-\sum_{\sigma<p\leqslant|\tau|^\delta}\log|\tau| + 4\sum_{\sigma<p\leqslant|\tau|^\delta}\log p\bigg\} \\ &\leqslant e^{-\{1/\delta-4+o(1)\}|\tau|^\delta}. \end{aligned}$$

2° Case of $c_1\sigma^{1/2}\log\sigma \leqslant |\tau| \leqslant \sigma^{1/\delta}$

For $p \geqslant \sigma^{1/2} \geqslant 2$, we can write

$$\begin{aligned}|E_p^*(s)| &\leqslant \int_0^{\pi/2} \big\{D_p(\theta)^{\sigma-1} + D_p(\pi-\theta)^{\sigma-1}\big\}\cos\theta\,\mathrm{d}\theta\\ &= \bigg\{1 + O\bigg(\frac{\sigma}{p^2}\bigg)\bigg\}\int_0^{\pi/2}\big(e^{2[(\sigma-1)/p]\cos\theta} + e^{-2[(\sigma-1)/p]\cos\theta}\big)\cos\theta\,\mathrm{d}\theta\end{aligned}$$

and

$$\begin{aligned}|E_p^*(\sigma)| &= \int_0^{\pi/2} \big\{D_p(\theta)^{\sigma-1} - D_p(\pi-\theta)^{\sigma-1}\big\}\cos\theta\,\mathrm{d}\theta\\ &= \bigg\{1 + O\bigg(\frac{\sigma}{p^2}\bigg)\bigg\}\int_0^{\pi/2}\big(e^{2[(\sigma-1)/p]\cos\theta} - e^{-2[(\sigma-1)/p]\cos\theta}\big)\cos\theta\,\mathrm{d}\theta.\end{aligned}$$

From these, we deduce that

$$\bigg|\frac{E_p^*(s)}{E_p^*(\sigma)}\bigg| \leqslant \bigg\{1 + O\bigg(\frac{\sigma}{p^2} + \frac{1}{e^{\sigma/p}}\bigg)\bigg\} \qquad (2 \leqslant \sigma^{1/2} \leqslant p \leqslant \sigma) \tag{5.5}$$

where we have used the following facts

$$\int_0^{\pi/2} e^{2[(\sigma-1)/p]\cos\theta}\cos\theta\,\mathrm{d}\theta \gg e^{\sigma/p} \qquad\text{and}\qquad \int_0^{\pi/2} e^{-2[(\sigma-1)/p]\cos\theta}\cos\theta\,\mathrm{d}\theta \ll 1.$$

Inserting (5.5) into (5.2), for $2 \leqslant \sigma^{1/2} \leqslant p \leqslant \sigma$ we obtain

$$\begin{aligned}\bigg|\frac{E_p(s)}{E_p(\sigma)}\bigg| &\leqslant \exp\bigg\{-\log\bigg|\frac{s-1}{\sigma-1}\bigg| + C\bigg(\frac{\sigma}{p^2} + \frac{1}{e^{\sigma/p}}\bigg)\bigg\}\\ &\leqslant \begin{cases} e^{-\tau^2/(2\sigma^2)+C\sigma/p^2+Ce^{-\sigma/p}} & \text{if } \ 3 \leqslant |\tau| \leqslant \sigma,\\ e^{-\frac12\log(1+\tau^2/\sigma^2)+C\sigma/p^2+Ce^{-\sigma/p}} & \text{if } \ \sigma \leqslant |\tau| \leqslant \sigma^{1/\delta},\end{cases}\end{aligned}$$

where $C > 0$ is an absolute constant.

Now by multiplying these inequalities for $\sigma/(4\log\sigma) \leqslant p \leqslant \sigma/(2\log\sigma)$ and the trivial inequality $|E_p(s)| \leqslant E_p(\sigma)$ for the other p, we get

$$\begin{aligned}\bigg|\frac{E(s,y)}{E(\sigma,y)}\bigg| &\leqslant \exp\bigg\{-\sum_{\sigma/(4\log\sigma)\leqslant p\leqslant\sigma/(2\log\sigma)}\bigg(\frac{\tau^2}{2\sigma^2} - \frac{C\sigma}{p^2} - \frac{C}{e^{\sigma/p}}\bigg)\bigg\}\\ &\leqslant \exp\bigg\{-\bigg(\frac{\tau^2}{16\sigma(\log\sigma)^2} - 10C - \frac{10C}{\sigma\log\sigma}\bigg)\bigg\}\\ &\leqslant \exp\bigg\{-\frac{c_2\tau^2}{\sigma(\log\sigma)^2}\bigg\}\end{aligned}$$

if $c_1\sigma^{1/2}\log\sigma \leqslant |\tau| \leqslant \sigma$, and

$$\begin{aligned}\bigg|\frac{E(s,y)}{E(\sigma,y)}\bigg| &\leqslant \exp\bigg\{-\sum_{\sigma/(4\log\sigma)\leqslant p\leqslant\sigma/(2\log\sigma)}\bigg[\frac12\log\bigg(1+\frac{\tau^2}{\sigma^2}\bigg) - \frac{C\sigma}{p^2} - \frac{C}{e^{\sigma/p}}\bigg]\bigg\}\\ &\leqslant \exp\bigg\{-\bigg[\frac{\sigma}{8\log\sigma}\log\bigg(1+\frac{\tau^2}{\sigma^2}\bigg) - 10C - \frac{10C}{\sigma\log\sigma}\bigg]\bigg\}\\ &\leqslant \exp\big\{-c_3|\tau|^\delta\big\}\end{aligned} \tag{5.6}$$

if $\sigma \leqslant |\tau| \leqslant \sigma^{1/\delta}$. This completes the proof. □

§ 6. Proof of Theorem 3

We follow the argument of Granville & Soundararajan [4] to prove Theorem 3. We shall divide the proof in several steps which are embodied in the following lemmas.

The first one is a classic integration formula (see [4], page 1019).

Lemma 6.1. *Let $c > 0$, $\lambda > 0$ and $N \in \mathbb{N}$. Then we have*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^s \Big(\frac{e^{\lambda s}-1}{\lambda s}\Big)^N \frac{\mathrm{d}s}{s} = \begin{cases} 0 & \text{if } 0 < y < e^{-\lambda N}, \\ \in [0,1] & \text{if } e^{-\lambda N} \leqslant y < 1, \\ 1 & \text{if } y \geqslant 1. \end{cases} \tag{6.1}$$

The second one is an analogue for (3.6) and (3.7) of [4] (see also Lemma 3.1 of [20]).

Lemma 6.2. *Let $t \geqslant 1$, $y \geqslant 2e^t$ and $0 < \lambda \leqslant e^{-t}$. Then we have*

$$\Phi(t,y) \leqslant \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s,y)}{(e^\gamma t)^{2s}}\frac{e^{\lambda s}-1}{\lambda s}\frac{\mathrm{d}s}{s} \leqslant \Phi(te^{-\lambda},y), \tag{6.2}$$

$$\Phi(te^{-\lambda},y) - \Phi(t,y) \leqslant \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s,y)}{(e^\gamma t)^{2s}}\frac{e^{\lambda s}-1}{\lambda s}\big(e^{2\lambda s} - e^{-2\lambda s}\big)\frac{\mathrm{d}s}{s}. \tag{6.3}$$

Proof. Denote by $\mathbf{1}_X(\omega)$ the characteristic function of the set $X \subset \Omega$. Then by Lemma 6.1 with $N = 1$ and $c = \kappa$, we have

$$\mathbf{1}_{\{\omega\in\Omega : L(1,g^\natural(\omega);y) > (e^\gamma t)^2\}}(\omega) \leqslant \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \Big(\frac{L(1,g^\natural(\omega);y)}{(e^\gamma t)^2}\Big)^s \frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s.$$

Integrating over Ω and interchanging the order of integrations yield

$$\begin{aligned}\Phi(t,y) &\leqslant \int_\Omega \bigg\{\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \Big(\frac{L(1,g^\natural(\omega);y)}{(e^\gamma t)^2}\Big)^s \frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s\bigg\}\,\mathrm{d}\mu(\omega) \\ &= \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s,y)}{(e^\gamma t)^{2s}}\frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s.\end{aligned}$$

This proves the first inequality of (6.2). The second can be treated by noticing that

$$\begin{aligned}\mathbf{1}_{\{\omega\in\Omega : L(1,g^\natural(\omega);y) > (e^{\gamma-\lambda} t)^2\}}(\omega) &= \mathbf{1}_{\{\omega\in\Omega : L(1,g^\natural(\omega);y) > (e^\gamma t)^2\}}(\omega) \\ &\quad + \mathbf{1}_{\{\omega\in\Omega : (e^\gamma t)^2 \geqslant L(1,g^\natural(\omega);y) > (e^{\gamma-\lambda} t)^2\}}(\omega) \\ &\geqslant \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \Big(\frac{L(1,g^\natural(\omega);y)}{(e^\gamma t)^2}\Big)^s \frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s.\end{aligned}$$

From (6.2), we can deduce

$$\begin{aligned}\Phi(te^{-\lambda},y) - \Phi(t,y) &\leqslant \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s,y)}{(e^{\gamma-\lambda} t)^{2s}}\frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s - \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s,y)}{(e^{\gamma+\lambda} t)^{2s}}\frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s,y)}{(e^\gamma t)^{2s}}\frac{e^{\lambda s}-1}{\lambda s^2}\big(e^{2\lambda s} - e^{-2\lambda s}\big)\,\mathrm{d}s.\end{aligned}$$

This completes the proof. □

Lemma 6.3. *Let $t \geqslant 1$, $y \geqslant 2e^t$ and $0 < \kappa\lambda \leqslant 1$. Then we have*

$$\frac{1}{2\pi i}\int_{\kappa-i\kappa}^{\kappa+i\kappa} \frac{E(s,y)}{(e^\gamma t)^{2s}}\frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s = \frac{E(\kappa,y)}{\kappa\sqrt{2\pi\sigma_2}(e^\gamma t)^{2\kappa}}\biggl\{1+O\biggl(\kappa\lambda+\frac{\log\kappa}{\kappa}\biggr)\biggr\}.$$

Proof. First in view of (4.24) we write, for $s = \kappa + i\tau$ and $|\tau| \leqslant \kappa$,

$$E(s,y) = \exp\biggl\{\sigma_0 + i\sigma_1\tau - \frac{\sigma_2}{2}\tau^2 - i\frac{\sigma_3}{6}\tau^3 + O\bigl(\sigma_4\tau^4\bigr)\biggr\}$$

and

$$\frac{e^{\lambda s}-1}{\lambda s^2} = \frac{1}{\kappa}\biggl\{1-\frac{i}{\kappa}\tau + O\biggl(\kappa\lambda+\frac{\tau^2}{\kappa^2}\biggr)\biggr\}.$$

Since $\sigma_1 = \log t + \gamma$, we have

$$\frac{E(s,y)}{(e^\gamma t)^{2s}}\frac{e^{\lambda s}-1}{\lambda s^2} = \frac{E(\kappa,y)}{\kappa(e^\gamma t)^{2\kappa}}e^{-(\sigma_2/2)\tau^2}\biggl\{1-\frac{i}{\kappa}\tau - i\frac{\sigma_3}{6}\tau^3 + O\bigl(R(\tau)\bigr)\biggr\}$$

with

$$R(\tau) := \kappa\lambda + \kappa^{-2}\tau^2 + \sigma_4\tau^4 + \sigma_3^2\tau^6.$$

Now we integrate the last expression over $|\tau| \leqslant \kappa$ to obtain

$$\frac{1}{2\pi i}\int_{\kappa-i\kappa}^{\kappa+i\kappa} \frac{E(s,y)}{(e^\gamma t)^{2s}}\frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s = \frac{E(\kappa,y)}{2\pi\kappa(e^\gamma t)^{2\kappa}}\int_{-\kappa}^{\kappa} e^{-(\sigma_2/2)\tau^2}\bigl\{1+O\bigl(R(\tau)\bigr)\bigr\}\,\mathrm{d}\tau, \tag{6.4}$$

where we have used the fact that the integrals involving $(i/\kappa)\tau$ and $(i\sigma_3/6)\tau^3$ vanish.

On the other hand, using lemmas 4.5 and 4.6 we have

$$\int_{-\kappa}^{\kappa} e^{-(\sigma_2/2)\tau^2}\,\mathrm{d}\tau = \sqrt{\frac{2\pi}{\sigma_2}}\biggl\{1+O\biggl(\exp\biggl\{-\frac{1}{2}\kappa^2\sigma_2\biggr\}\biggr)\biggr\},$$

$$\begin{aligned}\int_{\kappa-i\kappa}^{\kappa+i\kappa} e^{-(\sigma_2/2)\tau^2}R(\tau)\,\mathrm{d}\tau &\ll \frac{1}{\sqrt{\sigma_2}}\biggl(\kappa\lambda+\frac{1}{\kappa^2\sigma_2}+\frac{\sigma_3^2}{\sigma_2^3}+\frac{\sigma_4}{\sigma_2^2}\biggr)\\ &\ll \frac{1}{\sqrt{\sigma_2}}\biggl(\kappa\lambda+\frac{\log\kappa}{\kappa}\biggr).\end{aligned}$$

Inserting these into (6.4), we obtain the desired result. □

Lemma 6.4. *Let δ and c_3 be two constants determined by Lemma 5.1. Then we have*

$$\int_{\kappa\pm i\kappa}^{\kappa\pm i\infty} \frac{E(s,y)}{(e^\gamma t)^{2s}}\frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s \ll \frac{E(\kappa,y)}{\kappa\sqrt{\sigma_2}(e^\gamma t)^{2\kappa}}R_1, \tag{6.5}$$

$$\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s,y)}{(e^\gamma t)^{2s}}\frac{e^{\lambda s}-1}{\lambda s^2}\bigl(e^{2\lambda s}-e^{-2\lambda s}\bigr)\,\mathrm{d}s \ll \frac{E(\kappa,y)}{\kappa\sqrt{\sigma_2}(e^\gamma t)^{2\kappa}}R_2, \tag{6.6}$$

uniformly for $t \geqslant 1$, $y \geqslant 2e^t$, $\kappa \geqslant 2$ and $0 < \lambda\kappa \leqslant 1$, where

$$\begin{aligned}R_1 &:= \lambda^{-1}e^{-c_3\kappa^\delta} + \lambda^{-1}(\kappa/\log\kappa)^{1/2}y^{-1/\delta},\\ R_2 &:= \lambda\kappa(\log\kappa)^{1/2} + e^{-(c_3/2)\kappa^\delta} + \lambda^{-1}(\kappa/\log\kappa)^{1/2}y^{-1/\delta}.\end{aligned}$$

Proof. We split the integral in (6.5) into two parts according to $\kappa \leqslant |\tau| \leqslant y^{1/\delta}$ or $|\tau| \geqslant y^{1/\delta}$. Using Lemma 5.1 with $\sigma = \kappa$ and the inequality $(e^{\lambda s} - 1)/s^2 \ll 1/\tau^2$, the integral in (6.5) is

$$\ll \frac{E(\kappa, y)}{(e^\gamma t)^{2\kappa}\lambda}\bigg(\frac{e^{-c_3\kappa^\delta}}{\kappa} + \frac{1}{y^{1/\delta}}\bigg),$$

which implies (6.5), in view of Lemma 4.5 with $J = 1$.

Similarly we split the integral in (6.6) into four parts according to

$$|\tau| \leqslant c_1\kappa^{1/2}\log\kappa, \qquad c_1\kappa^{1/2}\log\kappa < |\tau| \leqslant \kappa, \qquad \kappa < |\tau| \leqslant y^{1/\delta}, \qquad |\tau| \geqslant y^{1/\delta}.$$

By Lemma 5.1 with $\sigma = \kappa$ and the inequalities

$$(e^{\lambda s} - 1)/\lambda s \ll \min\{1,\ 1/(\lambda|\tau|)\},$$
$$(e^{2\lambda s} - e^{-2\lambda s})/s \ll \min\{\lambda,\ 1/|\tau|\},$$

the integral in (6.6) is, as before,

$$\ll_\varepsilon \frac{E(\kappa, y)}{(e^\gamma t)^{2\kappa}}\Big(\lambda\kappa^{1/2}\log\kappa + e^{-c_3\kappa^\delta} + \lambda^{-1}y^{-1/\delta}\Big),$$

which implies (6.6), as before. □

Now we are ready to complete the proof of Theorem 3. Lemma 6.3 and (6.5) of Lemma 6.4 give

$$\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}\frac{E(s,y)}{(e^\gamma t)^{2s}}\frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s = \frac{E(\kappa, y)}{\kappa\sqrt{2\pi\sigma_2}(e^\gamma t)^{2\kappa}}\big\{1 + O\big(R'\big)\big\} \tag{6.7}$$

where

$$R' := \frac{\log\kappa}{\kappa} + \kappa\lambda + \frac{e^{-c_3\kappa^\delta} + (\kappa/\log\kappa)^{1/2}y^{-1/\delta}}{\lambda}.$$

Taking $\lambda = \kappa^{-2}$ and noticing $y \geqslant 2e^t \asymp \kappa$ and $1/\delta > 4$, we deduce

$$R' \ll t/e^t. \tag{6.8}$$

Combining (6.7) and (6.8) with (6.2), we obtain

$$\Phi(t, y) \leqslant \frac{E(\kappa, y)}{\kappa\sqrt{2\pi\sigma_2}(e^\gamma t)^{2\kappa}}\bigg\{1 + O\bigg(\frac{t}{e^t}\bigg)\bigg\} \leqslant \Phi(te^{-\lambda}, y) \tag{6.9}$$

uniformly for $t \geqslant 1$, $y \geqslant 2e^t$ and $0 < \lambda \leqslant e^{-t}$.

On the other hand, (6.3) of Lemma 6.2 and (6.6) of Lemma 6.4 imply

$$\begin{aligned}\Phi(te^{-\lambda}, y) - \Phi(t, y) &\ll \frac{E(\kappa, y)}{\kappa\sqrt{\sigma_2}(e^\gamma t)^{2\kappa}}\bigg(\lambda\kappa(\log\kappa)^{1/2} + \frac{(\kappa/\log\kappa)^{1/2}}{e^{c_3\kappa^\delta}} + \frac{(\kappa/\log\kappa)^{1/2}}{\lambda y^{1/\delta}}\bigg)\\ &\ll \frac{E(\kappa, y)}{\kappa\sqrt{\sigma_2}(e^\gamma t)^{2\kappa}}\bigg(\lambda\kappa(\log\kappa)^{1/2} + \frac{(\kappa/\log\kappa)^{1/2}}{e^{c_3\kappa^\delta}}\bigg)\end{aligned}$$

when $y^{-1/(2\delta)}\kappa^{-1/2}(\log\kappa)^{-1} \leqslant \lambda \leqslant \kappa^{-1}$. Since $\Phi(te^{-\lambda}, y) - \Phi(t, y)$ is a non-decreasing function of λ, we deduce

$$\Phi(te^{-\lambda}, y) - \Phi(t, y) \ll \frac{E(\kappa, y)}{\kappa\sqrt{\sigma_2}(e^\gamma t)^{2\kappa}}\bigg(\lambda\kappa(\log\kappa)^{1/2} + \frac{(\kappa/\log\kappa)^{1/2}}{e^{c_3\kappa^\delta}} + \frac{\kappa(\log\kappa)^{1/2}}{y^{1/(2\delta)}}\bigg) \tag{6.10}$$

uniformly for $t \geqslant 1$, $y \geqslant 2e^t$ and $0 < \lambda \leqslant e^{-t}$. Obviously the estimates (6.9) and (6.10) imply the desired result. This completes the proof of Theorem 3. $\square$

§ 7. Proof of Theorem 4

Using Lemmas 4.1 and 4.5, we can write

$$\begin{aligned}\frac{E(\kappa,y)}{\kappa\sqrt{2\pi\sigma_2}(e^\gamma t)^{2\kappa}} &= \exp\Big\{\phi(\kappa,y) - 2\kappa(\gamma+\log t) + O(\log\kappa)\Big\}\\ &= \exp\bigg\{\kappa\bigg(2\log_2\kappa - 2\log t + \sum_{j=1}^{J}\frac{b_{j,0}}{(\log\kappa)^j} + O_J\big(R_J(\kappa,y)\big)\bigg)\bigg\}.\end{aligned}$$

On the other hand, Lemma 4.3 and (1.17) imply that

$$2\log_2\kappa + 2\gamma + \sum_{j=1}^{J}\frac{b_{j,1}}{(\log\kappa)^j} + O_J\big(R_J(\kappa,y)\big) = 2(\log t+\gamma).$$

Combining these estimates, we can obtain

$$\frac{E(\kappa,y)}{\kappa\sqrt{2\pi\sigma_2}(e^\gamma t)^{2\kappa}} = \exp\bigg\{-\kappa\bigg[\sum_{j=1}^{J}\frac{b_{j,1}-b_{j,0}}{(\log\kappa)^j} + O_J\big(R_J(\kappa,y)\big)\bigg]\bigg\}.$$

In view of (1.21), (4.2) and (4.15), we have $b_{j,1} - b_{j,0} = a_j$. This completes the proof. $\square$

§ 8. Proof of Corollary 5

We first prove an asymptotic developpment of $\kappa(t,y)$ in t.

Lemma 8.1. *For each integer $J \geqslant 1$, there are computable constants $\gamma_0, \gamma_1, \dots, \gamma_J$ such that the asymptotic formula*

$$\kappa(t,y) = e^{t-\gamma_0}\bigg\{1 + \sum_{j=1}^{J}\frac{\gamma_j}{t^j} + O_J\big(R_J^*(t,y)\big)\bigg\} \tag{8.1}$$

holds uniformly for $t \geqslant 1$ and $y \geqslant 2e^t$, where

$$R_N^*(t,y) := \frac{1}{t^{N+1}} + \frac{e^t t}{y\log y}.$$

Further γ_0 is given by (1.24) and $\gamma_1 = -\frac{1}{8}b_{1,1}^2 - \frac{1}{4}b_{2,1}$.

Proof. By Lemma 4.3 and (1.17), we have

$$2\log t = 2\log_2\kappa + \sum_{j=1}^{J+1}\frac{b_{j,1}}{(\log\kappa)^j} + O_J\big(R_{J+1}(\kappa,y)\big), \tag{8.2}$$

where $R_J(\kappa, y)$ is defined as in (1.20). From (8.2), we easily deduce that

$$\begin{aligned} t &= (\log\kappa)\prod_{j=1}^{J+1}\exp\bigg\{\frac{b_{j,1}}{2(\log\kappa)^j}\bigg\}\exp\big\{O_J\big(R_{J+1}(\kappa,y)\big)\big\} \\ &= (\log\kappa)\prod_{j=1}^{J+1}\bigg\{\sum_{m_j=0}^{J+1}\frac{1}{m_j!}\bigg(\frac{b_{j,1}}{2(\log\kappa)^j}\bigg)^{m_j} + O_J\big(R_{J+1}(\kappa,y)\big)\bigg\}. \end{aligned}$$

Developping the product, we get

$$t = (\log\kappa)\bigg\{\sum_{j=0}^{J+1}\frac{b'_j}{(\log\kappa)^j} + O_J\big(R_{J+1}(\kappa,y)\big)\bigg\},$$

where

$$\begin{aligned} b'_j &:= \sum_{\substack{m_1\geqslant 0,\dots,m_{J+1}\geqslant 0 \\ m_1+2m_2+\cdots+(J+1)m_{J+1}=j}} \frac{b_{1,1}^{m_1}\cdots b_{J+1,1}^{m_{J+1}}}{(2m_1)!!\cdots(2m_{J+1})!!} \\ &= \sum_{\substack{m_1\geqslant 0,\dots,m_j\geqslant 0 \\ m_1+2m_2+\cdots+jm_j=j}} \frac{b_{1,1}^{m_1}\cdots b_{j,1}^{m_j}}{(2m_1)!!\cdots(2m_j)!!}. \end{aligned}$$

Since $b'_0 = 1$ and $b'_1 = b_{1,1}/2 = \gamma_0$, the preceeding asymptotic formula can be written as

$$t = \log\kappa + \gamma_0 + \sum_{j=1}^{J}\frac{b'_{j+1}}{(\log\kappa)^j} + O_J\big(R_J^*(t,y)\big), \tag{8.3}$$

where we have used the fact that $\kappa(t,y) \asymp e^t$ (see Lemma 2.3) and $(\log k)R_{J+1}(\kappa,y) \asymp R_J^*(t,y)$.

With the help of (8.3), a simple recurrence argument shows that there are constants γ'_n such that

$$t = \log\kappa + \sum_{j=0}^{J}\frac{\gamma'_j}{t^j} + O_J\big(R_J^*(t,y)\big). \tag{8.4}$$

In fact taking $J = 0$ in (8.3), we see that (8.4) holds for $J = 0$. Suppose that it holds for $0, \dots, J-1$, i.e.

$$t = \log\kappa + \sum_{i=0}^{J-j-1}\frac{\gamma'_i}{t^i} + O\big(R_{J-j-1}^*(t,y)\big) \qquad (j = 0, \dots, J-1),$$

which is equivalent to

$$\log\kappa = t\bigg\{1 - \sum_{i=1}^{J-j}\frac{\gamma'_{i-1}}{t^i} + O\bigg(\frac{R_{J-j-1}^*(t,y)}{t}\bigg)\bigg\} \qquad (j = 0, \dots, J-1). \tag{8.5}$$

This holds also for $j = J$ if we use the convention:

$$\sum_{i=0}^{-1} = 0 \qquad \text{and} \qquad R_{-1}^*(t,y) := 1,$$

since $\log\kappa = t + O(1)$. Inserting it into (8.3), we easily see that (8.4) holds also for J. In particular we have

$$\gamma_1' = b_2' = \tfrac{1}{8}b_{1,1}^2 + \tfrac{1}{4}b_{2,1}.$$

Now (8.1) is an immediate consequence of (8.4) with

$$\gamma_j := \sum_{\substack{m_1\geqslant 0,\dots,m_J\geqslant 0\\ m_1+2m_2+\cdots+Jm_J=j}} (-1)^{m_1+\cdots+m_J}\frac{{\gamma_1'}^{m_1}\cdots{\gamma_J'}^{m_J}}{m_1!\cdots m_J!}.$$

This completes the proof. □

Now we are ready to prove Corollary 5.

Using (8.5), we have

$$\begin{aligned}\sum_{j=1}^{J}\frac{a_j}{(\log\kappa)^j} &= \sum_{j=1}^{J}\frac{a_j}{t^j}\bigg\{1-\sum_{i=1}^{J-j}\frac{\gamma_{i-1}'}{t^i}+O_N\bigg(\frac{R_{J-j-1}^*(t,y)}{t}\bigg)\bigg\}^{-j}\\ &= \sum_{j=1}^{J}\frac{\rho_j}{t^j}+O_J\bigg(\frac{R_{J-2}^*(t,y)}{t^2}\bigg),\end{aligned}\tag{8.6}$$

where the ρ_n are constants. In particular we have $\rho_1 = a_1 = 1$ and $\rho_2 = \gamma_0 + a_2$.

Now Theorem 4, (8.1) and (8.6) imply the result of Corollary with

$$a_1^* = \rho_1 = 1, \qquad a_j^* = \rho_j + \sum_{i=1}^{j-1}\gamma_i\rho_{j-i} \quad (j\geqslant 2).$$

This completes the proof of Corollary 5. □

§ 9. Proof of Theorem 2

For each $\eta\in(0,\frac{1}{2})$, define

$$\mathrm{H}_k^+(1;\eta) := \big\{f\in\mathrm{H}_k^*(1) : L(s,f)\neq 0,\ s\in\mathcal{S}\big\},$$

where $\mathcal{S} := \{s := \sigma+i\tau :\ \sigma\geqslant 1-\eta,\ |\tau|\leqslant 100k^\eta\}\cup\{s:=\sigma+i\tau :\ \sigma\geqslant 1,\ \tau\in\mathbb{R}\}$, and

$$\mathrm{H}_k^-(1;\eta) := \mathrm{H}_k^*(1)\smallsetminus\mathrm{H}_k^+(1;\eta).$$

Then we have (see [10], (1.11))

$$\big|\mathrm{H}_k^-(1;\eta)\big| \ll_\eta k^{31\eta}.\tag{9.1}$$

Our starting point in the proof of Theorem 2 is the evaluation of the moments of $L(1,f)$. For this, we recall a particular case of Proposition 6.1 of [10].

Lemma 9.1. *Let $\eta \in (0, \frac{1}{31})$ be fixed. There are two positive constants $c_i = c_i(\eta)$ $(i = 4, 5)$ such that*

$$\sum_{f \in \mathrm{H}_k^+(1;\eta)} \omega_f L(1, f)^s = E(s) + O_\eta\big(e^{-c_4 \log k / \log_2 k}\big) \tag{9.2}$$

uniformly for

$$k \geqslant 16, \qquad 2 \mid k \qquad \text{and} \qquad |s| \leqslant 2T_k \tag{9.3}$$

with

$$T_k := c_5 \log k / (\log_2 k \log_3 k).$$

Here $E(s)$ is defined by (1.15).

Let $\kappa(t, y)$ be the saddle-point determined by (1.17) and $\kappa_t := \kappa(t, \infty)$. For $k \geqslant 16, 2 \mid k$, $\lambda > 0$, $N \in \mathbb{N}$ and $t > 0$, introduce the two integrals

$$I_1(k, t; \lambda, N) := \frac{1}{2\pi i} \int_{\kappa_t - i\infty}^{\kappa_t + i\infty} \sum_{f \in \mathrm{H}_k^+(1;\eta)} \omega_f \left(\frac{L(1, f)}{(e^\gamma t)^2}\right)^s \left(\frac{e^{\lambda s} - 1}{\lambda s}\right)^{2N} \frac{\mathrm{d}s}{s}$$

and

$$I_2(k, t; \lambda, N) := \frac{1}{2\pi i} \int_{\kappa_t - i\infty}^{\kappa_t + i\infty} \frac{E(s)}{(e^\gamma t)^{2s}} \left(\frac{e^{\lambda s} - 1}{\lambda s}\right)^{2N} \frac{\mathrm{d}s}{s}.$$

Lemma 9.2. *Let $\eta \in (0, \frac{1}{200}]$ be fixed. Then we have*

$$\widetilde{F}_k(t) + O_\eta\big(k^{-5/6}\big) \leqslant I_1(k, t; \lambda, N) \leqslant \widetilde{F}_k(te^{-\lambda N}) + O_\eta\big(k^{-5/6}\big), \tag{9.4}$$

$$\Phi(t) \leqslant I_2(k, t; \lambda, N) \leqslant \Phi(te^{-\lambda N}) \tag{9.5}$$

uniformly for $k \geqslant 16$, $2 \mid k$, $\lambda > 0$, $N \in \mathbb{N}$ and $t > 0$. The implied constants depend on η only.

Proof. By exchanging the order of sommation and by using Lemma 6.1 with $c = \kappa_t$, we obtain

$$\begin{aligned} I_1(k, t; \lambda, N) &= \sum_{f \in \mathrm{H}_k^+(1;\eta)} \frac{\omega_f}{2\pi i} \int_{\kappa_t - i\infty}^{\kappa_t + i\infty} \left(\frac{L(1, f)}{(e^\gamma t)^2}\right)^s \left(\frac{e^{\lambda s} - 1}{\lambda s}\right)^{2N} \frac{\mathrm{d}s}{s}, \\ &\geqslant \sum_{f \in \mathrm{H}_k^+(1;\eta),\ L(1,f) \geqslant (e^\gamma t)^2} \omega_f. \end{aligned}$$

In view of the second estimate of (1.7) and of (9.1), we reintroduce the missing forms

$$\begin{aligned} I_1(k, t; \lambda, N) &\geqslant \sum_{f \in \mathrm{H}_k^*(1),\ L(1,f) \geqslant (e^\gamma t)^2} \omega_f + O\Big(\sum_{f \in \mathrm{H}_k^* \smallsetminus \mathrm{H}_k^+(1;\eta)} \omega_f\Big) \\ &\geqslant \sum_{f \in \mathrm{H}_k^*(1),\ L(1,f) \geqslant (e^\gamma t)^2} \omega_f + O\big(k^{-1+31\eta} \log k\big). \end{aligned}$$

Clearly this implies the first inequality of (9.4), thanks to (1.6) and (1.7).

Similarly, using Lemma 6.1 with $c = \kappa_t$, we find

$$\begin{aligned} I_1(k, t; \lambda, N) &\leqslant \sum_{\substack{f \in \mathrm{H}_k^+(1;\eta) \\ L(1,f) \geqslant (e^\gamma t)^2}} \omega_f + \sum_{\substack{f \in \mathrm{H}_k^+(1;\eta) \\ (e^\gamma t e^{-\lambda N})^2 \leqslant L(1,f) < (e^\gamma t)^2}} \omega_f \\ &= \sum_{\substack{f \in \mathrm{H}_k^+(1;\eta) \\ L(1,f) \geqslant (e^\gamma t e^{-\lambda N})^2}} \omega_f. \end{aligned}$$

As before, we can easily show that the last sum is $\leqslant \widetilde{F}_k(te^{-\lambda N}) + O\big(k^{-5/6}\big)$.

The estimates (9.5) can be proved in the same way as (6.2). □

Lemma 9.3. *Let $\eta \in (0, \frac{1}{200}]$ be fixed and c_4 be the positive constant given by Lemma 9.1. Then we have*

$$\begin{aligned}|I_1(k,t;\lambda,N) - I_2(k,t;\lambda,N)| &\ll e^{-c_4(\log k)/\log_2 k}\frac{(1+e^{\lambda\kappa_t})^{2N}\log T_k}{(e^\gamma t)^{2\kappa_t}}\\ &\quad + \frac{E(\kappa_t)+e^{-c_4(\log k)/\log_2 k}}{N(e^\gamma t)^{2\kappa_t}}\biggl(\frac{1+e^{\lambda\kappa_t}}{\lambda T_k}\biggr)^{2N}\end{aligned} \tag{9.6}$$

uniformly for $\lambda > 0$, $N \in \mathbb{N}$, $k \geqslant 16$, $2 \mid k$ and $t \leqslant T(k)$, where $T(k)$ is given by (1.10). The implied constant depends on η only.

Proof. By the definitions of I_1 and I_2, we can write

$$\begin{aligned}&I_1(k,t;\lambda,N) - I_2(k,t;\lambda,N)\\ &= \frac{1}{2\pi i}\int_{\kappa_t - i\infty}^{\kappa_t+i\infty}\biggl(\sum_{f\in \mathrm{H}_k^+(1;\eta)}\omega_f L(1,f)^s - E(s)\biggr)\biggl(\frac{e^{\lambda s}-1}{\lambda s}\biggr)^{2N}\frac{\mathrm{d}s}{s(e^\gamma t)^{2s}}.\end{aligned}$$

In order to estimate the last integral, we split it into two parts according to $|\tau| \leqslant T_k$ or $|\tau| > T_k$.

In view of (1.18), it is easy to see that $\kappa_t \leqslant T_k$ for $t \leqslant T(k)$. Thus we may apply (9.2) of Lemma 9.1 for $s = \kappa_t + i\tau$ with $|\tau| \leqslant T_k$. Note that $|(e^{\lambda s}-1)/(\lambda s)| \leqslant 1 + e^{\lambda\kappa_t}$ for $s = \kappa_t + i\tau$, which is easily seen by looking at the cases $|\lambda s| \leqslant 1$ and $|\lambda s| > 1$. The contribution of $|\tau| \leqslant T_k$ to $|I_1(k,t;\lambda,N) - I_2(k,t;\lambda,N)|$ is

$$\ll e^{-c_4(\log k)/\log_2 k}\frac{(1+e^{\lambda\kappa_t})^{2N}\log T_k}{(e^\gamma t)^{2\kappa_t}}. \tag{9.7}$$

Since $\kappa_t \leqslant T_k$ for $t \leqslant T(k)$, we can apply (9.2) of Lemma 9.1 to write, for $s = \kappa_t + i\tau$ with $\tau \in \mathbb{R}$,

$$\begin{aligned}\biggl|\sum_{f\in \mathrm{H}_k^+(1;\eta)}\omega_f L(1,f)^s - E(s)\biggr| &\leqslant \sum_{f\in \mathrm{H}_k^+(1;\eta)}\omega_f L(1,f)^{\kappa_t} + E(\kappa_t)\\ &\leqslant 2E(\kappa_t) + O\bigl(e^{-c_4(\log k)/\log_2 k}\bigr).\end{aligned}$$

Thus the contribution of $|\tau| > T_k$ to $|I_1(k,t;\lambda,N) - I_2(k,t;\lambda,N)|$ is

$$\begin{aligned}&\ll \frac{E(\kappa_t)+e^{-c_4(\log k)/\log_2 k}}{(e^\gamma t)^{2\kappa_t}}\int_{|\tau|\geqslant T_k}\biggl(\frac{1+e^{\lambda\kappa_t}}{\lambda|\tau|}\biggr)^{2N}\frac{\mathrm{d}\tau}{|\tau|}\\ &\ll \frac{E(\kappa_t)+e^{-c_4(\log k)/\log_2 k}}{N(e^\gamma t)^{2\kappa_t}}\biggl(\frac{1+e^{\lambda\kappa_t}}{\lambda T_k}\biggr)^{2N}.\end{aligned} \tag{9.8}$$

Combining (9.7) and (9.8) yields to the required estimate. □

End of the proof of Theorem 2

For simplicity of notation, we write

$$I_j := I_j(k,t;\lambda,N) \qquad \text{and} \qquad I_j^+ := I_j(k, te^{\lambda N};\lambda,N) \qquad (j=1,2).$$

By using Lemma 9.2, we have

$$\begin{aligned}\widetilde{F}_k(t) &\leqslant I_1 + O\bigl(k^{-5/6}\bigr)\\ &= I_2 + O\bigl(|I_1 - I_2| + k^{-5/6}\bigr)\\ &\leqslant \Phi(te^{-\lambda N}) + O\bigl(|I_1-I_2| + k^{-5/6}\bigr)\\ &\leqslant \Phi(t) + |\Phi(te^{-\lambda N}) - \Phi(t)| + O\bigl(|I_1-I_2| + k^{-5/6}\bigr)\end{aligned} \tag{9.9}$$

and

$$
\begin{aligned}
\widetilde{F}_k(t) &\geqslant I_1^+ + O\big(k^{-5/6}\big) \\
&= I_2^+ + O\big(|I_1^+ - I_2^+| + k^{-5/6}\big) \\
&\geqslant \Phi(te^{\lambda N}) + O\big(|I_1^+ - I_2^+| + k^{-5/6}\big) \\
&\geqslant \Phi(t) - |\Phi(t) - \Phi(te^{\lambda N})| + O\big(|I_1^+ - I_2^+| + k^{-5/6}\big).
\end{aligned}
\tag{9.10}
$$

In view of (6.10) and Theorem 3, we have

$$\big|\Phi(t) - \Phi(te^{-\lambda N})\big| \ll \Phi(t)\big\{\lambda N \kappa_t (\log \kappa_t)^{1/2} + e^{-(c_3/2)\kappa_t^\delta}\big\}$$

for $\lambda N \leqslant e^{-t}$. Take

$$\lambda = e^{5A}/T_k \qquad \text{and} \qquad N = [\log_2 k]. \tag{9.11}$$

Since $T_k = e^{T(k)+\frac{3}{2}\log_3 k + 2C + \log c_5}$, it is easy to see that

$$\lambda N \leqslant e^{-T(k)-2C} T(k)^{-1/2} \qquad \text{and} \qquad \kappa_t \asymp e^t.$$

Inserting these estimates into the preceeding inequality, a simple calculation shows that

$$\big|\Phi(t) - \Phi(te^{-\lambda N})\big| \leqslant \Phi(t)\big\{e^{t-T(k)-C}(t/T(k))^{1/2} + O\big(e^{-c_6 e^{\delta t}}\big)\big\}, \tag{9.12}$$

provided the constant C is suitably large, where $c_6 = c_6(\eta, \delta)$ is a positive constant.

Similarly by using (6.10) with $te^{\lambda N}$ in place of t, we have

$$\big|\Phi(t) - \Phi(te^{\lambda N})\big| \ll \Phi(te^{\lambda N})\big\{\lambda N \kappa_{te^{\lambda N}} (\log \kappa_{te^{\lambda N}})^{1/2} + e^{-(c_3/2)\kappa_{te^{\lambda N}}^\delta}\big\}.$$

Since for $t \leqslant T(k)$ we have

$$te^{\lambda N} = t + O\big((\log_2 k)^3 (\log_3 k)/\log k\big) \qquad \text{and} \qquad \kappa_{te^{\lambda N}} \asymp e^{te^{\lambda N}} \asymp e^t,$$

the preceeding estimate can be writen as

$$
\begin{aligned}
\big|\Phi(t) - \Phi(te^{\lambda N})\big| &\leqslant \tfrac{1}{4}\Phi(te^{\lambda N})\big\{e^{t-T(k)-C}(t/T(k))^{1/2} + O\big(e^{-c_6 e^{\delta t}}\big)\big\} \\
&\leqslant \tfrac{1}{4}\Phi(t)\big\{e^{t-T(k)-C}(t/T(k))^{1/2} + O\big(e^{-c_6 e^{\delta t}}\big)\big\} \\
&\quad + \tfrac{1}{4}\big|\Phi(t) - \Phi(te^{\lambda N})\big|\big\{e^{t-T(k)-C}(t/T(k))^{1/2} + O\big(e^{-c_6 e^{\delta t}}\big)\big\},
\end{aligned}
$$

from which we deduce that

$$\big|\Phi(t) - \Phi(te^{\lambda N})\big| \leqslant \Phi(t)\big\{e^{t-T(k)-C}(t/T(k))^{1/2} + O\big(e^{-c_6 e^{\delta t}}\big)\big\}. \tag{9.13}$$

By using Lemma 9.3 with $te^{\lambda N}$ in place of t, we have

$$
\begin{aligned}
|I_1^+ - I_2^+| &\ll e^{-c_4(\log k)/\log_2 k} \frac{(1 + e^{\lambda \kappa_{te^{\lambda N}}})^{2N} \log T_k}{(e^\gamma te^{\lambda N})^{2\kappa_{te^{\lambda N}}}} \\
&\quad + \frac{E(\kappa_{te^{\lambda N}}) + e^{-c_4(\log k)/\log_2 k}}{N(e^\gamma te^{\lambda N})^{2\kappa_{te^{\lambda N}}}} \bigg(\frac{1 + e^{\lambda \kappa_{te^{\lambda N}}}}{\lambda T_k}\bigg)^{2N}.
\end{aligned}
$$

On the other hand, by using Theorem 3 and (1.25), it is easy to see that there is a positive constant c such that

$$\Phi(te^{\lambda N}) \asymp \Phi(t) \sim \frac{E(\kappa_t)}{\kappa_t\sqrt{2\pi\sigma_2}(e^{\gamma}t)^{2\kappa_t}} \gg e^{-c_8 e^t/t} \gg e^{-c_9(\log k)/[(\log_2 k)^{7/2}\log_3 k]}$$

for $t \leqslant T(k)$. Thanks to Lemma 4.5, the previous estimate can be written as

$$|I_1^+ - I_2^+| \ll \Phi(t)\frac{1}{N}\left(\frac{\kappa_{te^{\lambda N}}}{\log \kappa_{te^{\lambda N}}}\right)^{1/2}\left(\frac{1+e^{\lambda\kappa_{te^{\lambda N}}}}{\lambda T_k}\right)^{2N} \ll \frac{\Phi(t)}{(\log k)^A}. \tag{9.14}$$

Similarly we can prove (even more easily)

$$|I_1 - I_2| \ll \Phi(t)/(\log k)^A. \tag{9.15}$$

Inserting (9.12) and (9.16) into (9.9) and (9.13) and (9.15) into (9.10), we obtain

$$\widetilde{F}_k(t) \leqslant \Phi(t)\big\{1 + e^{t-T(k)-C}(t/T(k))^{1/2} + O\big(e^{-c_6 e^{\delta t}} + (\log k)^{-A}\big)\big\}$$

and

$$\widetilde{F}_k(t) \geqslant \Phi(t)\big\{1 - e^{t-T(k)-C}(t/T(k))^{1/2} + O\big(e^{-c_6 e^{\delta t}} + (\log k)^{-A}\big)\big\}.$$

This implies the first asymptotic formula of (1.13) by taking $\eta = \frac{1}{200}$ and $\delta = \frac{1}{5}$.

The second can be established similarly. This completes the proof of Theorem 2. □

§ 10. Proof of Theorem 1

The formula (1.9) is an immediate consequence of Theorem 2 and (1.25).

Taking $t = T(k)$ in (1.9), we find that

$$e^{-c_1'(\log k)/\{(\log_2 k)^{7/2}\log_3 k\}} \ll \widetilde{F}_k(T(k)) \ll e^{-c_2'(\log k)/\{(\log_2 k)^{7/2}\log_3 k\}}, \tag{10.1}$$

where c_1' and c_2' are two positive constants. Clearly (10.1) and (1.8) imply (1.11).

The related results on $\widetilde{G}_k(t)$ and $G_k(T(k))$ can be proved similarly. This completes the proof of Theorem 1. □

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